"JUST THE MATHS"

SLIDES NUMBER

9.6

MATRICES 6
(Eigenvalues and eigenvectors)

by

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UNIT 9.6 - MATRICES 6

EIGENVALUES AND EIGENVECTORS

9.6.1 THE STATEMENT OF THE PROBLEM

Let A be any square matrix, and let X be a column vector with the same number of rows as there are columns in A.

For example, if A is of order $m \times m$, then X must be of order $m \times 1$ and AX will also be of order $m \times 1$.

We ask the question:

"Is it ever possible that AX can be just a scalar multiple of X ?"

We exclude the case when the elements of X are all zero.

ILLUSTRATIONS

1.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ -3 \end{bmatrix}.$$

2.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The formal statement of the problem

For a given square matrix, A, we investigate the existence of column vectors, X, such that

$$AX = \lambda X,$$

for some scalar quantity λ.

Each such column vector is called an **"eigenvector"** of the matrix, A.

Each corresponding value of λ is called an **"eigenvalue"** of the matrix, A.

Notes:

(i) The German word "eigen" means"hidden".

(ii) Other alternative names are "latent values and latent vectors" or "characteristic values and characteristic vectors".

(iii) In the discussion which follows, A will be, mostly, a matrix of order 3×3.

9.6.2 THE SOLUTION OF THE PROBLEM

Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then the matrix equation, $AX = \lambda X$, means

$$\begin{aligned} a_1x + b_1y + c_1z &= \lambda x, \\ a_2x + b_2y + c_2z &= \lambda y, \\ a_3x + b_3y + c_3z &= \lambda z; \end{aligned}$$

or, on rearrangement,

$$\begin{aligned} (a_1 - \lambda)x + b_1y + c_1z &= 0, \\ a_2x + (b_2 - \lambda)y + c_2z &= 0, \\ a_3x + b_3y + (c_3 - \lambda)z &= 0. \end{aligned}$$

This is a set of homogeneous linear equations in x, y and z and may be written

$$(A - \lambda I)X = [0],$$

where I denotes the identity matrix of order 3×3.

From Unit 7.4, the three homogeneous linear equations have a solution other than $x = 0$, $y = 0$, $z = 0$ if

$$|\mathrm{A} - \lambda \mathrm{I}| = \begin{vmatrix} a_1 - \lambda & b_1 & c_1 \\ a_2 & b_2 - \lambda & c_2 \\ a_3 & b_3 & c_3 - \lambda \end{vmatrix} = 0.$$

On expansion, this gives a cubic equation in λ called the **"characteristic equation"** of A.

The left-hand side of the characteristic equation is called the **"characteristic polynomial"** of A.

The characteristic equation of a 3×3 matrix, being a <u>cubic</u> equation, will (in general) have three solutions.

EXAMPLES

1. Determine the eigenvalues and eigenvectors of the matrix

$$\mathrm{A} = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}.$$

Solution

(a) The eigenvalues

The characteristic equation is given by

$$0 = \begin{vmatrix} 2 - \lambda & 4 \\ 5 & 3 - \lambda \end{vmatrix} = \lambda^2 - 5\lambda - 14 = (\lambda + 2)(\lambda - 7).$$

The eigenvalues are therefore $\lambda = -2$ and $\lambda = 7$.

(b) The eigenvectors

Case 1. $\lambda = -2$

We require to solve the equation $x + y = 0$,

giving $x : y = -1 : 1$ and a corresponding eigenvector

$$X = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where α is any **non-zero** scalar.

Case 2. $\lambda = 7$

We require to solve the equation $5x - 4y = 0$,

giving $x : y = 4 : 5$ and a corresponding eigenvector

$$X = \beta \begin{bmatrix} 4 \\ 5 \end{bmatrix},$$

where β is any **non-zero** scalar.

2. Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

Solution

(a) The eigenvalues

The characteristic equation is given by

$$0 = |A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}.$$

Direct expansion of the determinant gives the equation

$$-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0,$$

which will factorise into

$$(1+\lambda)^2(8-\lambda) = 0.$$

Note:

Students who have studied row and column operations for determinants (see Unit 7.3) may obtain this by simplifying the determinant first.

One way is to subtract the third column from the first column and then add the third row to the first row.

The eigenvalues are therefore $\lambda = -1$(repeated) and $\lambda = 8$.

(b) The eigenvectors

Case 1. $\lambda = 8$

We solve the homogeneous equations

$$\begin{aligned} -5x + 2y + 4z &= 0, \\ 2x - 8y + 2z &= 0, \\ 4x + 2y - 5z &= 0. \end{aligned}$$

Eliminating x from the second and third equations gives $18y - 9z = 0$.

Eliminating y from the second and third equations gives $18x - 18z = 0$.

Since z appears twice, we may let $z = 1$ to give $y = \frac{1}{2}$ and $x = 1$ and, hence,

$$x : y : z = 1 : \frac{1}{2} : 1 = 2 : 1 : 2$$

The eigenvectors corresponding to $\lambda = 8$ are

$$\mathrm{X} = \alpha \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix},$$

where α is any **non-zero** scalar.

Case 2. $\lambda = -1$

We solve the homogeneous equations

$$\begin{aligned} 4x + 2y + 4z &= 0, \\ 2x + y + 2z &= 0, \\ 4x + 2y + 4z &= 0. \end{aligned}$$

These are all the same equation, $2x + y + 2z = 0$.

First form of solution

Two of the variables may be chosen at random (say $y = \beta$ and $z = \gamma$).

Then the third variable may be expressed in terms of them; (in this case $x = -\frac{1}{2}\beta - \gamma$).

Neater form of solution

First obtain a pair of independent particular solutions by setting two of the variables, in turn, at 1 and another at 0.

For example $y = 1$ and $z = 0$ gives $x = -\frac{1}{2}$ while $y = 0$ and $z = 1$ gives $x = -1$.

Using the particular solutions $x = -\frac{1}{2}$, $y = 1$, $z = 0$ and $x = -1$, $y = 0$, $z = 1$ the general solution is given by

$$\mathrm{X} = \beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \alpha \text{ and } \beta \text{ not both zero.}$$

Notes:

(i) Similar results in Case 2 could be obtained by choosing a **different** pair of the three variables at random.

(ii) Other special cases arise if the three homogeneous equations reduce to a single equation in which one or even two of the variables is absent.

ILLUSTRATIONS

1. If the homogeneous equations reduced to $y = 0$, the corresponding eigenvectors could be given by

$$\mathrm{X} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

   This allows $x = \alpha$ and $z = \gamma$ to be chosen at random, assuming that α and γ are not both zero simultaneously.

2. If the homogeneous equations reduced to

$$3x + 5z = 0,$$

then the corresponding eigenvectors could be given by

$$\mathrm{X} = \alpha \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{5} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

This allows $x = \alpha$ and $y = \beta$ to be chosen at random, assuming that α and β are not both zero simultaneously.