Use de Moivre's Theorem to show that

$$\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^7 \left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^5 = -i. \qquad \textit{(6 marks)}$$

2	$(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^7 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$	B1		
	$(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})^5$			
	$= \cos\frac{5\pi}{3} - i\sin\frac{5\pi}{3}$	B1		
	Expansion of			Or
	$= (\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6})(\cos\frac{5\pi}{3} - i\sin\frac{5\pi}{3})$	M1		$\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ M1A1
				$-\frac{\sqrt{3}}{4} - \frac{3}{4}i - \frac{1}{4}i + \frac{\sqrt{3}}{4}$ A1
				$-i$ A1
	$= \cos\left(\frac{7\pi}{6} - \frac{5\pi}{3}\right) + i\sin\left(\frac{7\pi}{6} - \frac{5\pi}{3}\right)$	A1		Allow sign error
	$= \cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})$	A1		
	$= -i$	A1	6	AG
	Total		**6**	

(a) (i) Verify that $z = 2e^{\frac{1}{4}\pi i}$ is a root of the equation $z^4 = -16$. *(1 mark)*

(ii) Find the other three roots of this equation, giving each root in the form $re^{i\theta}$, where r is real and $-\pi < \theta \leqslant \pi$. *(3 marks)*

(iii) Illustrate the four roots of the equation by points on an Argand diagram. *(2 marks)*

(b) (i) Show that

$$(z - 2e^{\frac{1}{4}\pi i})(z - 2e^{-\frac{1}{4}\pi i}) = z^2 - 2\sqrt{2}\,z + 4.$$ *(3 marks)*

(ii) Express $z^4 + 16$ as the product of two quadratic factors with real coefficients. *(3 marks)*

Q	Solution	Marks	Total	Comments
6 (a)(i)	$\left(2e^{\frac{\pi i}{4}}\right)^4 = 16e^{\pi i} = -16$	B1	1	
	$z = 2e^{\left(\frac{\pi i}{4}+\frac{2k\pi i}{4}\right)}$	M1		
	$k=0,\ z=2e^{\frac{\pi i}{4}}$			
	other roots, $z = 2e^{-\pi i/4}$, $z = 2e^{\pm 3\pi i/4}$	A2,1,0	3	Allow if quoted correctly Deduct A1 for answers outside range indicated
(iii)	Argand diagram: $r = 2$ Properly spaced	B1 B1	2	CAO except for $r = 2$
(b)(i)	$\left(z - 2e^{\frac{\pi i}{4}}\right)\left(z - 2e^{-\frac{\pi i}{4}}\right)$			
	$= z^2 - 2\left(e^{\frac{\pi i}{4}} + e^{-\frac{\pi i}{4}}\right)z + 4e^{\frac{\pi i}{4}}e^{-\frac{\pi i}{4}}$	M1		
	$= z^2 - 2\times2\cos\frac{\pi}{4}z + 4$	A1		Must see some working for this A1
	$= z^2 - 2\sqrt{2}z + 4$	A1	3	AG
(ii)	$(z-2e^{3\pi i/4})(z-2e^{-3\pi i/4})$			
	$= z^2 - 2\times2\cos\frac{3\pi}{4}z + 4 = z^2 + 2\sqrt{2}z + 4$	M1A1		
	$z^4 + 16 = (z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2}z + 4)$	A1	3	If quoted allow B1
	Total		**12**	

(a) (i) Use de Moivre's theorem to show that

$$(\cos\theta + \mathrm{i}\sin\theta)^4 + (\cos\theta - \mathrm{i}\sin\theta)^4 = 2\cos 4\theta.$$ *(2 marks)*

(ii) Deduce that

$$(\cot\theta + \mathrm{i})^4 + (\cot\theta - \mathrm{i})^4 = \frac{2\cos 4\theta}{\sin^4\theta}, \quad \theta \neq r\pi.$$ *(1 mark)*

(b) Verify that $\cot\frac{1}{8}\pi$ is a root of

$$(z + \mathrm{i})^4 + (z - \mathrm{i})^4 = 0$$

and find the **three** other roots of this equation giving each answer in the form $+\cot\alpha$ or $-\cot\alpha$, where $0 < \alpha \leqslant \frac{\pi}{2}$. *(4 marks)*

(c) Express the equation in part (b) in the form

$$z^4 + bz^2 + c = 0,$$

where b and c are real numbers to be determined. *(2 marks)*

(d) Hence, or otherwise, find in surd form the value of $\cot^2\frac{\pi}{8}$. *(3 marks)*

Q	Solution	Marks	Total	Comments
6 (a)(i)				
	$(\cos\theta + \text{i}\sin\theta)^4 + (\cos\theta - \text{i}\sin\theta)^4$			
	$= \cos 4\theta + \text{i}\sin 4\theta + \cos 4\theta - \text{i}\sin 4\theta$	M1		
	$= 2\cos 4\theta$	A1	2	
(ii)	$\div \sin^4\theta$			
	$(\cot\theta + \text{i})^4 + (\cot\theta - \text{i})^4 = \dfrac{2\cos 4\theta}{\sin^4\theta}$	B1	1	
(b)	$\cot\theta$ is a root of			
	$(z+\text{i})^4 + (z-\text{i})^4 = 0$ if $\cos 4\theta = 0$	M1		or substitute in
	$4\theta = \frac{\pi}{2}$, $\theta = \frac{\pi}{8} \therefore \cot\frac{\pi}{8}$ is a root	A1		
	Other roots are when $4\theta = -\frac{\pi}{2} \pm \frac{3\pi}{2}$	M1		
	$\theta = -\frac{\pi}{8}, \pm\frac{3\pi}{8}$			
	$\therefore$ roots are $\pm\cot\frac{\pi}{8}, \pm\cot\frac{3\pi}{8}$	A1	4	
(c)	$z^4 - 6z^2 + 1 = 0$	M1A1	2	M0 if no binomial coefficients
(d)	$z^2 = 3 \pm 2\sqrt{2}$	M1A1		
	$\cot\frac{\pi}{8} > 1$, $\cot^2\frac{\pi}{8} = 3 + 2\sqrt{2}$	E1	3	
	Total		**12**	

It is given that

$$w = \frac{1}{\sqrt{2}}(-1 + \mathrm{i}).$$

(a) (i) Show that $|w| = 1$.

(ii) Express w in the form $\mathrm{e}^{\mathrm{i}\theta}$ where $-\pi < \theta \leqslant \pi$. *(3 marks)*

(b) Solve $z^3 = w$, giving your answers in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$. *(4 marks)*

(c) (i) Show that

$$(1 - w)(1 - w^*) = 2 + \sqrt{2},$$

where w^* is the complex conjugate of w. *(3 marks)*

(ii) The sum of the geometric series $\sum_{r=0}^{11} w^r$ is S.

Show that

$$S = \frac{2}{1 - w}$$

and hence express S in the form $1 + p\mathrm{i}$, where p is real. *(5 marks)*

Q	Solution	Marks	Total	Comments
6 (a)(i)	$\lvert w\rvert=\left\{\left(-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2\right\}^{\frac{1}{2}}=1$	M1A1		M1 for finding either $\lvert w\rvert$ or $\arg w$
(ii)	$\arg w=\frac{3\pi}{4}$	A1	3	
(b)	$z=e^{\frac{\pi i}{4}+2k\frac{\pi i}{3}}\qquad k=-1,0,1$	M1A1F		allow B1F for one correct root with no method shown
	$=e^{\frac{\pi i}{4}},\ e^{\frac{11\pi i}{12}},\ e^{\frac{-5\pi i}{12}}$	A1		any correct root
		A1	4	other two correct
(c)(i)	$(1-w)(1-w^*)$			
	$=\left(1-e^{\frac{3\pi i}{4}}\right)\left(1-e^{\frac{-3\pi i}{4}}\right)$	M1		Alternative method: $\left(1-\frac{1}{\sqrt{2}}(-1+i)\right)\left(1-\frac{1}{\sqrt{2}}(-1-i)\right)$ M1
	$=1+1-\left(e^{\frac{3\pi i}{4}}+e^{\frac{-3\pi i}{4}}\right)$	A1		Multiplied out (any form) A1 $2+\sqrt{2}$ A1
	$=2-2\cos\frac{3\pi}{4}$			
	$=2+\sqrt{2}$	A1	3	
(ii)	$\sum_{r=0}^{11}w^r=\frac{1-w^{12}}{1-w}$	M1		
	$=\frac{1-e^{9\pi i}}{1-w}$	A1		
	$=\frac{2}{1-w}$	A1		
	Real part is 1 shown	B1		
	Imaginary part $\frac{\sqrt{2}}{2+\sqrt{2}}i$ $\left(=1+\left(\sqrt{2}-1\right)i\right)$	B1	5	accept any form
	Total		**15**	