FP2 – De Moivre's Theorem

Challenge 1

Use de Moivre's Theorem to show that

$$\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)^7 \left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)^5 = -\mathrm{i}.$$

(6 marks)

Challenge 2

(a) (i) Verify that $z = 2e^{\frac{1}{4}\pi i}$ is a root of the equation $z^4 = -16$. *(1 mark)*

(ii) Find the other three roots of this equation, giving each root in the form $re^{i\theta}$, where r is real and $-\pi < \theta \leqslant \pi$. *(3 marks)*

(iii) Illustrate the four roots of the equation by points on an Argand diagram. *(2 marks)*

(b) (i) Show that

$$(z - 2e^{\frac{1}{4}\pi i})(z - 2e^{-\frac{1}{4}\pi i}) = z^2 - 2\sqrt{2}\,z + 4.$$ *(3 marks)*

(ii) Express $z^4 + 16$ as the product of two quadratic factors with real coefficients. *(3 marks)*

Challenge 3

(a) (i) Use de Moivre's theorem to show that

$$(\cos\theta + \mathrm{i}\sin\theta)^4 + (\cos\theta - \mathrm{i}\sin\theta)^4 = 2\cos 4\theta. \qquad \textit{(2 marks)}$$

(ii) Deduce that

$$(\cot\theta + \mathrm{i})^4 + (\cot\theta - \mathrm{i})^4 = \frac{2\cos 4\theta}{\sin^4\theta}, \quad \theta \neq r\pi. \qquad \textit{(1 mark)}$$

(b) Verify that $\cot\frac{1}{8}\pi$ is a root of

$$(z+\mathrm{i})^4 + (z-\mathrm{i})^4 = 0$$

and find the **three** other roots of this equation giving each answer in the form $+\cot\alpha$ or $-\cot\alpha$, where $0 < \alpha \leqslant \frac{\pi}{2}$. *(4 marks)*

(c) Express the equation in part (b) in the form

$$z^4 + bz^2 + c = 0,$$

where b and c are real numbers to be determined. *(2 marks)*

(d) Hence, or otherwise, find in surd form the value of $\cot^2\frac{\pi}{8}$. *(3 marks)*

Final Challenge

It is given that

$$w = \frac{1}{\sqrt{2}}(-1 + \mathrm{i}).$$

(a) (i) Show that $|w| = 1$.

(ii) Express w in the form $\mathrm{e}^{\mathrm{i}\theta}$ where $-\pi < \theta \leqslant \pi$. *(3 marks)*

(b) Solve $z^3 = w$, giving your answers in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$. *(4 marks)*

(c) (i) Show that

$$(1 - w)(1 - w^*) = 2 + \sqrt{2},$$

where w^* is the complex conjugate of w. *(3 marks)*

(ii) The sum of the geometric series $\sum_{r=0}^{11} w^r$ is S.

Show that

$$S = \frac{2}{1 - w}$$

and hence express S in the form $1 + p\mathrm{i}$, where p is real. *(5 marks)*