7 The matrix **M** is defined by

$$\mathbf{M} = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}.$$

(a) Show that **M** has just two eigenvalues, 1 and 3. *(6 marks)*

(b) Find an eigenvector corresponding to each eigenvalue. *(5 marks)*

The matrix **M** represents a linear transformation, T, of three dimensional space.

(c) Write down a vector equation of the line of invariant points of T. *(1 mark)*

(d) Write down a vector equation of another line which is invariant under T. *(1 mark)*

Q	Solution	Marks	Total	Comments
7 (a)	$\begin{vmatrix} 2-\lambda & -1 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 1 & 2-\lambda \end{vmatrix}$	M1		
	$(2-\lambda)(6-5\lambda+\lambda^2-2)+1(2-2\lambda)+1(-1+\lambda)$	M1		a.e.f.
	$-\lambda^3+7\lambda^2-15\lambda+9$	A1		
	$(\lambda-1)(\lambda-3)^2=0$	M1A1		
	$\lambda=1$ or 3	A1	6	a.g.
(b)	$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}\mathbf{v}=0,\quad \begin{bmatrix} -1 & -1 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}\mathbf{v}=0$	M1,M1		
	E.g. $\mathbf{v}=\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},\quad \mathbf{v}=\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$	M1A1,A1	5	M1A1 for either
(c)	$\mathbf{r}=\lambda\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$	B1√	1	
(d)	$\mathbf{r}=\mu\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$	B1√	1	
	Total		**13**	

5 It is given that the transformation represented by the matrix

$$\mathbf{M} = \begin{bmatrix} 6 & -6 & 1 \\ 3 & -3 & 1 \\ 2 & -4 & 3 \end{bmatrix}$$

has invariant lines

$$x = y = z,$$
$$x = y = \tfrac{1}{2}z,$$
$$x = 2y,\ z = 0.$$

(a) (i) Write down an eigenvector corresponding to each invariant line. *(2 marks)*

(ii) Hence find the eigenvalues of **M**. *(3 marks)*

(b) Write down a matrix **U** and a diagonal matrix **D** such that

$$\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}.$$ *(3 marks)*

5 (a)(i)	$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}$	B2	2	B1 for any 1
(ii)	$\mathbf{M}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow \lambda = 1$			
	$\mathbf{M}\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}2\\2\\4\end{bmatrix} \Rightarrow \lambda = 2$			
	$\mathbf{M}\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}6\\3\\0\end{bmatrix} \Rightarrow \lambda = 3$	B3	3	Allow B1 for each eigenvalue
(b)	$\mathbf{U} = \begin{bmatrix}1 & 1 & 2\\1 & 1 & 1\\1 & 2 & 0\end{bmatrix}$	B1√		
	$\mathbf{D} = \begin{bmatrix}1 & & \\ & 2 & \\ & & 3\end{bmatrix}$	B1√ B1	3	If orders correspond
	Total		**8**	

6 (a) Find the characteristic equation of the matrix

$$\mathbf{M} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix},$$

and hence find the eigenvalues of **M**, and corresponding eigenvectors, $\mathbf{v}_1$ and $\mathbf{v}_2$. *(7 marks)*

(b) (i) Express $\begin{bmatrix} 5 \\ -7 \end{bmatrix}$ in the form $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2$, where α and β are real numbers. *(3 marks)*

(ii) Hence find $\mathbf{M}^{-1}\begin{bmatrix} 5 \\ -7 \end{bmatrix}$. *(3 marks)*

Q	Solution	Marks	Total	Comments
6 (a)	$\begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} = 0$	M1 A1		
	$(3-\lambda)(4-\lambda) - 2 = 0$			
	$\lambda^2 - 7\lambda + 10 = 0 \Rightarrow \lambda = 2, 5$	M1A1		
	$\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$	M1A1A1	7	
(b)(i)	$\begin{bmatrix} 5 \\ -7 \end{bmatrix} = 4\begin{bmatrix} 2 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	M1A2	3	
(ii)	$\mathbf{M}^{-1}\begin{bmatrix} 5 \\ -7 \end{bmatrix} = \frac{4}{2}\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \frac{3}{5}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	M1A1		
	$\frac{1}{5}\begin{bmatrix} 17 \\ -13 \end{bmatrix}$	A1	3	
	Total		**13**	

6 The matrix $\mathbf{M}$ is defined by

$$\mathbf{M} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}.$$

(a) Verify that $\mathbf{M}$ has an eigenvalue 1 and find a corresponding eigenvector. *(5 marks)*

(b) (i) Show that $\mathbf{M}$ has only one other distinct eigenvalue and find this eigenvalue. *(4 marks)*

(ii) Deduce that any non-zero vector of the form $\begin{bmatrix} p \\ q \\ q \end{bmatrix}$, where p and q are real, is an eigenvector corresponding to this eigenvalue. *(3 marks)*

(c) (i) Find a Cartesian equation of a line l_1 lying in the plane $x = 0$ such that each point of l_1 is invariant under the transformation T represented by $\mathbf{M}$. *(2 marks)*

(ii) Find a Cartesian equation of another line l_2 in the plane $x = 0$ which is invariant under T. *(2 marks)*

Q	Solution	Marks	Total	Comments
6 (a)	$\begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix} = 0$	M1A1		
	$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$			
	$\therefore 3x = 0$ $y + 2z = 0$	M1A1		
	eigenvector is $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$	A1F	5	OE
(b)(i)	$\begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 1 & 3-\lambda \end{vmatrix}$			
	$= (4-\lambda)((2-\lambda)(3-\lambda)-2)$	M1A1		Allow whenever this line appears
	$= (4-\lambda)(\lambda-4)(\lambda-1)$	A1F		Provided quadratic factorises
	$\lambda = 4$	A1	4	
(ii)	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2y+2z \\ y-z \end{bmatrix}$	M1		Accept $\begin{bmatrix} p \\ q \\ q \end{bmatrix}$ substituted in and verified
	$y = z$, x any value	A1		
	eigenvector $\begin{bmatrix} p \\ q \\ q \end{bmatrix}$	A1	3	AG
(c)(i)	$x = 0, y = -2z$ $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -2t \\ t \end{bmatrix}$			
	$\therefore$ point invariant	M1A1	2	
(ii)	$x = 0, y = z$ $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = 4\begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$			
	$\therefore$ invariant line	M1A1	2	
	Total		**16**	

4 The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{bmatrix} 5 & -2 \\ 12 & -5 \end{bmatrix}.$$

(a) (i) Find the eigenvalues of $\mathbf{M}$ and corresponding eigenvectors. *(7 marks)*

(ii) Write down a matrix $\mathbf{U}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{M} = \mathbf{U\,D\,U}^{-1}.$$ *(2 marks)*

(iii) Evaluate $\mathbf{D}^2$. *(1 mark)*

(b) (i) Show that $\mathbf{M}^n = \mathbf{U\,D}^n\,\mathbf{U}^{-1}$, where n is positive. *(2 marks)*

(ii) Hence find $\mathbf{M}^{10}$ and $\mathbf{M}^{11}$. *(3 marks)*

Q	Solution	Marks	Total	Comments
4 (a)(i)	$\begin{vmatrix} 5-\lambda & -2 \\ 12 & -5-\lambda \end{vmatrix}$	M1		
	$(5-\lambda)(-5-\lambda)+24$	A1		
	$\lambda^2-1=0$	A1F		
	$\lambda=\pm 1$	A1F		
	$\lambda=1,\ \begin{bmatrix} 4 & -2 \\ 12 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix},\ 2x-y=0$	M1		Must have consistent equations
	eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	A1		no ft
	$\lambda=-1,\ \begin{bmatrix} 6 & -2 \\ 12 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix},\ 3x-y=0$			
	eigenvector is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$	A1	7	no ft
(ii)	$\mathbf{U}=\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ $\mathbf{D}=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	B1FB1F	2	ft on incorrect eigenvalues and eigenvectors; must be consistent for both B1
(iii)	$\mathbf{D}^2=\mathbf{I}$	B1F	1	
(b)(i)	$\mathbf{M}^n=\mathbf{U\,D\,U}^{-1}\ \mathbf{U\,D\,U}^{-1}\ldots(n \text{ times})$	B1		
	Use of $\mathbf{U}^{-1}\mathbf{U}=\mathbf{I}$	B1	2	at any stage
	$\mathbf{M}^n=\mathbf{U\,D}^n\,\mathbf{U}^{-1}$			AG
(ii)	$\mathbf{M}^{10}=\mathbf{U\,D}^{10}\,\mathbf{U}^{-1}$	M1		could give M1 for $\mathbf{U\,D}^{11}\,\mathbf{U}^{-1}=\mathbf{M}^{11}$
	$=\mathbf{I}$	A1		
	$\mathbf{M}^{11}=\mathbf{U\,D\,U}^{-1}=\mathbf{M}$	A1	3	
	Total		**15**	