FP4 Eigenvectors Challenge

Challenge 1

It is given that the transformation represented by the matrix

$$\mathbf{M} = \begin{bmatrix} 6 & -6 & 1 \\ 3 & -3 & 1 \\ 2 & -4 & 3 \end{bmatrix}$$

has invariant lines

$$x = y = z,$$
$$x = y = \tfrac{1}{2}z,$$
$$x = 2y,\ z = 0.$$

(a) (i) Write down an eigenvector corresponding to each invariant line. *(2 marks)*

(ii) Hence find the eigenvalues of **M**. *(3 marks)*

(b) Write down a matrix **U** and a diagonal matrix **D** such that

$$\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}.$$ *(3 marks)*

Challenge 2

(a) Find the characteristic equation of the matrix

$$\mathbf{M} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix},$$

and hence find the eigenvalues of **M**, and corresponding eigenvectors, $\mathbf{v}_1$ and $\mathbf{v}_2$.

(7 marks)

(b) (i) Express $\begin{bmatrix} 5 \\ -7 \end{bmatrix}$ in the form $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2$, where α and β are real numbers. *(3 marks)*

(ii) Hence find $\mathbf{M}^{-1}\begin{bmatrix} 5 \\ -7 \end{bmatrix}$. *(3 marks)*

Challenge 3

The matrix **M** is defined by

$$\mathbf{M} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}.$$

(a) Verify that **M** has an eigenvalue 1 and find a corresponding eigenvector. *(5 marks)*

(b) (i) Show that **M** has only one other distinct eigenvalue and find this eigenvalue. *(4 marks)*

(ii) Deduce that any non-zero vector of the form $\begin{bmatrix} p \\ q \\ q \end{bmatrix}$, where p and q are real, is an eigenvector corresponding to this eigenvalue. *(3 marks)*

(c) (i) Find a Cartesian equation of a line l_1 lying in the plane $x = 0$ such that each point of l_1 is invariant under the transformation T represented by **M**. *(2 marks)*

(ii) Find a Cartesian equation of another line l_2 in the plane $x = 0$ which is invariant under T. *(2 marks)*

Final Challenge

The matrix **M** is given by

$$\mathbf{M} = \begin{bmatrix} 5 & -2 \\ 12 & -5 \end{bmatrix}.$$

(a) (i) Find the eigenvalues of **M** and corresponding eigenvectors. *(7 marks)*

(ii) Write down a matrix **U** and a diagonal matrix **D** such that

$$\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}.$$ *(2 marks)*

(iii) Evaluate $\mathbf{D}^2$. *(1 mark)*

(b) (i) Show that $\mathbf{M}^n = \mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$, where n is positive. *(2 marks)*

(ii) Hence find $\mathbf{M}^{10}$ and $\mathbf{M}^{11}$. *(3 marks)*