Further Pure 2

Hyperbolic Functions

Name:

January 2006

7 (a) Use the definitions

$$\sinh\theta = \frac{1}{2}(e^{\theta} - e^{\ \theta}) \qquad \text{and} \qquad \cosh\theta = \frac{1}{2}(e^{\theta} + e^{\ \theta})$$

to show that:

(i) $2\sinh\theta\cosh\theta = \sinh 2\theta$; *(2 marks)*

(ii) $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$. *(3 marks)*

(b) A curve is given parametrically by

$$x = \cosh^3\theta, \quad y = \sinh^3\theta$$

(i) Show that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{9}{4}\sinh^2 2\theta\cosh 2\theta$$ *(6 marks)*

June 2006

3 The curve C has equation

$$y = \cosh x - 3\sinh x$$

(a) (i) The line $y = -1$ meets C at the point $(k, -1)$.

Show that

$$e^{2k} - e^{k} - 2 = 0$$ *(3 marks)*

(ii) Hence find k, giving your answer in the form $\ln a$. *(4 marks)*

(b) (i) Find the x-coordinate of the point where the curve C intersects the x-axis, giving your answer in the form $p\ln a$. *(4 marks)*

(ii) Show that C has no stationary points. *(3 marks)*

(iii) Show that there is exactly one point on C for which $\frac{d^2y}{dx^2} = 0$. *(1 mark)*

January 2007

1 (a) Given that

$$4\cosh^2 x = 7\sinh x + 1$$

find the two possible values of $\sinh x$. *(4 marks)*

(b) Hence obtain the two possible values of x, giving your answers in the form $\ln p$. *(3 marks)*

4 (a) Given that $y = \operatorname{sech} t$, show that:

(i) $\dfrac{dy}{dt} = -\operatorname{sech} t \tanh t$; *(3 marks)*

(ii) $\left(\dfrac{dy}{dt}\right)^2 = \operatorname{sech}^2 t - \operatorname{sech}^4 t$. *(2 marks)*

June 2007

7 A curve has equation $y = 4\sqrt{x}$.

(a) Show that the length of arc s of the curve between the points where $x = 0$ and $x = 1$ is given by

$$s = \int_0^1 \sqrt{\frac{x+4}{x}}\, dx$$ *(4 marks)*

(b) (i) Use the substitution $x = 4\sinh^2\theta$ to show that

$$\int \sqrt{\frac{x+4}{x}}\, dx = \int 8\cosh^2\theta\ d\theta$$ *(5 marks)*

(ii) Hence show that

$$s = 4\sinh^{-1} 0.5 + \sqrt{5}$$ *(6 marks)*

January 2008

7 (a) Given that $y = \ln \tanh \frac{x}{2}$, where $x > 0$, show that

$$\frac{dy}{dx} = \operatorname{cosech} x$$ *(6 marks)*

(b) A curve has equation $y = \ln \tanh \frac{x}{2}$, where $x > 0$. The length of the arc of the curve between the points where $x = 1$ and $x = 2$ is denoted by s.

(i) Show that

$$s = \int_1^2 \coth x \, dx$$ *(2 marks)*

(ii) Hence show that $s = \ln(2\cosh 1)$. *(4 marks)*

June 2008

1 (a) Express

$$5\sinh x + \cosh x$$

in the form $Ae^{x} + Be^{-x}$, where A and B are integers. *(2 marks)*

(b) Solve the equation

$$5\sinh x + \cosh x + 5 = 0$$

giving your answer in the form $\ln a$, where a is a rational number. *(4 marks)*

5 (a) Use the definition $\cosh x = \frac{1}{2}(e^{x} + e^{-x})$ to show that $\cosh 2x = 2\cosh^2 x - 1$. *(2 marks)*

January 2009

1 (a) Use the definitions $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$ and $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$ to show that

$$1 + 2\sinh^2\theta = \cosh 2\theta$$ *(3 marks)*

(b) Solve the equation

$$3\cosh 2\theta = 2\sinh\theta + 11$$

giving each of your answers in the form $\ln p$. *(6 marks)*

2 (a) Indicate on an Argand diagram the region for which $|z - 4i| \leqslant 2$. *(4 marks)*

(b) The complex number z satisfies $|z - 4i| \leqslant 2$. Find the range of possible values of $\arg z$. *(4 marks)*

5 (a) Given that $u = \cosh^2 x$, show that $\frac{du}{dx} = \sinh 2x$. *(2 marks)*

(b) Hence show that

$$\int_0^1 \frac{\sinh 2x}{1 + \cosh^4 x}\,dx = \tan^{-1}(\cosh^2 1) - \frac{\pi}{4}$$ *(5 marks)*

7 (a) Show that

$$\frac{d}{dx}\left(\cosh^{-1}\frac{1}{x}\right) = \frac{-1}{x\sqrt{1-x^2}}$$ *(3 marks)*

(b) A curve has equation

$$y = \sqrt{1-x^2} - \cosh^{-1}\frac{1}{x} \qquad (0 < x < 1)$$

Show that:

(i) $\frac{dy}{dx} = \frac{\sqrt{1-x^2}}{x}$; *(4 marks)*

June 2009

4 (a) Sketch the graph of $y = \tanh x$. *(2 marks)*

(b) Given that $u = \tanh x$, use the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x} to show that

$$x = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$$ *(6 marks)*

(c) (i) Show that the equation

$$3\,\text{sech}^2 x + 7\tanh x = 5$$

can be written as

$$3\tanh^2 x - 7\tanh x + 2 = 0$$ *(2 marks)*

(ii) Show that the equation

$$3\tanh^2 x - 7\tanh x + 2 = 0$$

has only one solution for x.

Find this solution in the form $\frac{1}{2}\ln a$, where a is an integer. *(5 marks)*

January 2010

1 (a) Use the definitions $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$ to show that

$$\cosh^2 x - \sinh^2 x = 1$$ *(3 marks)*

(b) (i) Express

$$5\cosh^2 x + 3\sinh^2 x$$

in terms of $\cosh x$. *(1 mark)*

(ii) Sketch the curve $y = \cosh x$. *(1 mark)*

(iii) Hence solve the equation

$$5\cosh^2 x + 3\sinh^2 x = 9.5$$

giving your answers in logarithmic form. *(4 marks)*

4 A curve C is given parametrically by the equations

$$x = \frac{1}{2}\cosh 2t, \qquad y = 2\sinh t$$

(a) Express

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

in terms of $\cosh t$. *(6 marks)*

June 2010

1 (a) Show that

$$9\sinh x - \cosh x = 4e^x - 5e^{-x}$$ *(2 marks)*

(b) Given that

$$9\sinh x - \cosh x = 8$$

find the exact value of $\tanh x$. *(7 marks)*

5 (a) Using the identities

$$\cosh^2 t - \sinh^2 t = 1, \quad \tanh t = \frac{\sinh t}{\cosh t} \quad \text{and} \quad \operatorname{sech} t = \frac{1}{\cosh t}$$

show that:

(i) $\tanh^2 t + \operatorname{sech}^2 t = 1$; *(2 marks)*

(ii) $\frac{d}{dt}(\tanh t) = \operatorname{sech}^2 t$; *(3 marks)*

(iii) $\frac{d}{dt}(\operatorname{sech} t) = -\operatorname{sech} t \tanh t$. *(3 marks)*

January 2011

4 (a) Prove that the curve

$$y = 12\cosh x - 8\sinh x - x$$

has exactly one stationary point. *(7 marks)*

(b) Given that the coordinates of this stationary point are (a, b), show that $a + b = 9$. *(4 marks)*

6 (a) Given that

$$x = \ln(\sec t + \tan t) - \sin t$$

show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sin t \tan t$$ *(4 marks)*

June 2011

2 (a) Use the definitions of $\cosh\theta$ and $\sinh\theta$ in terms of e^{θ} to show that

$$\cosh x \cosh y - \sinh x \sinh y = \cosh(x - y)$$ *(4 marks)*

(b) It is given that x satisfies the equation

$$\cosh(x - \ln 2) = \sinh x$$

(i) Show that $\tanh x = \frac{5}{7}$. *(4 marks)*

(ii) Express x in the form $\frac{1}{2}\ln a$. *(2 marks)*

January 2012

1 (a) Show, by means of a sketch, that the curves with equations

$$y = \sinh x$$

and $$y = \operatorname{sech} x$$

have exactly one point of intersection. *(4 marks)*

(b) Find the x-coordinate of this point of intersection, giving your answer in the form $a \ln b$. *(4 marks)*

3 A curve has cartesian equation

$$y = \frac{1}{2}\ln(\tanh x)$$

(a) Show that

$$\frac{dy}{dx} = \frac{1}{\sinh 2x}$$ *(4 marks)*

June 2012

1 (a) Sketch the curve $y = \cosh x$. *(1 mark)*

(b) Solve the equation

$$6\cosh^2 x - 7\cosh x - 5 = 0$$

giving your answers in logarithmic form. *(6 marks)*

6 (a) Show that

$$\frac{1}{4}(\cosh 4x + 2\cosh 2x + 1) = \cosh^2 x \cosh 2x$$ *(3 marks)*

(b) Show that, if $y = \cosh^2 x$, then

$$1 + \left(\frac{dy}{dx}\right)^2 = \cosh^2 2x$$ *(3 marks)*

January 2013

1 (a) Show that

$$12\cosh x - 4\sinh x = 4\mathrm{e}^{x} + 8\mathrm{e}^{-x}$$ *(2 marks)*

(b) Solve the equation

$$12\cosh x - 4\sinh x = 33$$

giving your answers in the form $k\ln 2$. *(5 marks)*

5 (a) Using the definition $\tanh y = \dfrac{\mathrm{e}^{y} - \mathrm{e}^{-y}}{\mathrm{e}^{y} + \mathrm{e}^{-y}}$, show that, for $|x| < 1$,

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$ *(3 marks)*

(b) Hence, or otherwise, show that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tanh^{-1} x) = \dfrac{1}{1-x^2}$. *(3 marks)*

(c) Use integration by parts to show that

$$\int_0^{\frac{1}{2}} 4\tanh^{-1} x \,\mathrm{d}x = \ln\left(\frac{3^m}{2^n}\right)$$

where m and n are positive integers. *(5 marks)*

June 2013

2 (a) (i) Sketch on the axes below the graphs of $y = \sinh x$ and $y = \cosh x$. *(3 marks)*

(ii) Use your graphs to explain why the equation

$$(k + \sinh x)\cosh x = 0$$

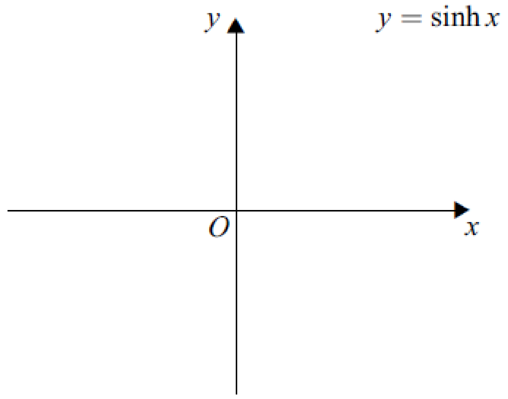
where k is a constant, has exactly one solution. *(1 mark)*

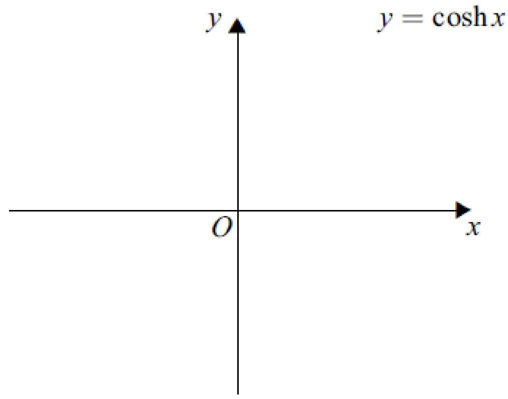
(b) A curve C has equation $y = 6\sinh x + \cosh^2 x$. Show that C has only one stationary point and show that its y-coordinate is an integer. *(5 marks)*

JESTION PART FERENCE

Answer space for question 2

a)(i)





6 (a) Show that $\dfrac{1}{5\cosh x - 3\sinh x} = \dfrac{e^x}{m + e^{2x}}$, where m is an integer. *(3 marks)*

(b) Use the substitution $u = e^x$ to show that

$$\int_0^{\ln 2} \frac{1}{5\cosh x - 3\sinh x}\,dx = \frac{\pi}{8} - \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$$ *(5 marks)*

7 (a) (i) Show that

$$\frac{d}{du}\left(2u\sqrt{1+4u^2} + \sinh^{-1} 2u\right) = k\sqrt{1+4u^2}$$

where k is an integer. *(4 marks)*

(ii) Hence show that

$$\int_0^1 \sqrt{1+4u^2}\,du = p\sqrt{5} + q\sinh^{-1} 2$$

where p and q are rational numbers. *(2 marks)*