# Further Pure 2

## Length of a curve and Area of Revolution

Name:

January 2006

**7** (a) Use the definitions

$$\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta}) \qquad \text{and} \qquad \cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$$

to show that:

(i) $2\sinh\theta\cosh\theta = \sinh 2\theta$; *(2 marks)*

(ii) $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$. *(3 marks)*

(b) A curve is given parametrically by

$$x = \cosh^3\theta, \quad y = \sinh^3\theta$$

(i) Show that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{9}{4}\sinh^2 2\theta\cosh 2\theta$$ *(6 marks)*

(ii) Show that the length of the arc of the curve from the point where $\theta = 0$ to the point where $\theta = 1$ is

$$\frac{1}{2}\left[(\cosh 2)^{\frac{3}{2}} - 1\right]$$ *(6 marks)*

June 2006

**2** A curve has parametric equations

$$x = t - \frac{1}{3}t^3, \quad y = t^2$$

(a) Show that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 + t^2)^2$$ *(3 marks)*

(b) The arc of the curve between $t = 1$ and $t = 2$ is rotated through $2\pi$ radians about the $x$-axis.

Show that $S$, the surface area generated, is given by $S = k\pi$, where $k$ is a rational number to be found. *(5 marks)*

January 2007

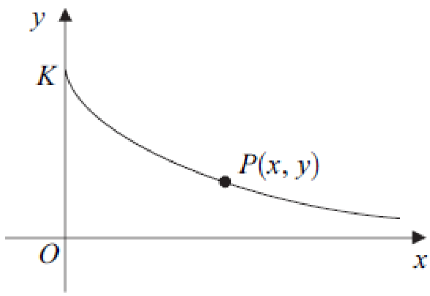
**4** (a) Given that $y = \operatorname{sech} t$, show that:

(i) $\dfrac{dy}{dt} = -\operatorname{sech} t \tanh t$; *(3 marks)*

(ii) $\left(\dfrac{dy}{dt}\right)^2 = \operatorname{sech}^2 t - \operatorname{sech}^4 t$. *(2 marks)*

(b) The diagram shows a sketch of part of the curve given parametrically by

$$x = t - \tanh t \qquad y = \operatorname{sech} t$$




The curve meets the $y$-axis at the point $K$, and $P(x, y)$ is a general point on the curve. The arc length $KP$ is denoted by $s$. Show that:

(i) $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \tanh^2 t$; *(4 marks)*

(ii) $s = \ln \cosh t$; *(3 marks)*

(iii) $y = e^{-s}$. *(2 marks)*

(c) The arc $KP$ is rotated through $2\pi$ radians about the $x$-axis. Show that the surface area generated is

$$2\pi(1 - e^{-s})$$ *(4 marks)*

June 2007

**7** A curve has equation $y = 4\sqrt{x}$.

(a) Show that the length of arc $s$ of the curve between the points where $x = 0$ and $x = 1$ is given by

$$s = \int_0^1 \sqrt{\frac{x+4}{x}}\, dx$$ *(4 marks)*

(b) (i) Use the substitution $x = 4\sinh^2 \theta$ to show that

$$\int \sqrt{\frac{x+4}{x}}\, dx = \int 8\cosh^2 \theta \, d\theta$$ *(5 marks)*

(ii) Hence show that

$$s = 4\sinh^{-1} 0.5 + \sqrt{5}$$ *(6 marks)*

January 2008

7 (a) Given that $y = \ln \tanh \frac{x}{2}$, where $x > 0$, show that

$$\frac{dy}{dx} = \operatorname{cosech} x$$ *(6 marks)*

(b) A curve has equation $y = \ln \tanh \frac{x}{2}$, where $x > 0$. The length of the arc of the curve between the points where $x = 1$ and $x = 2$ is denoted by $s$.

(i) Show that

$$s = \int_1^2 \coth x \, dx$$ *(2 marks)*

(ii) Hence show that $s = \ln(2\cosh 1)$. *(4 marks)*

June 2008

5 (a) Use the definition $\cosh x = \frac{1}{2}(e^x + e^{-x})$ to show that $\cosh 2x = 2\cosh^2 x - 1$. *(2 marks)*

(b) (i) The arc of the curve $y = \cosh x$ between $x = 0$ and $x = \ln a$ is rotated through $2\pi$ radians about the $x$-axis. Show that $S$, the surface area generated, is given by

$$S = 2\pi \int_0^{\ln a} \cosh^2 x \, dx$$ *(3 marks)*

(ii) Hence show that

$$S = \pi\left(\ln a + \frac{a^4 - 1}{4a^2}\right)$$ *(5 marks)*

January 2009

7 (a) Show that

$$\frac{d}{dx}\left(\cosh^{-1}\frac{1}{x}\right) = \frac{-1}{x\sqrt{1-x^2}}$$ *(3 marks)*

(b) A curve has equation

$$y = \sqrt{1-x^2} - \cosh^{-1}\frac{1}{x} \qquad (0 < x < 1)$$

Show that:

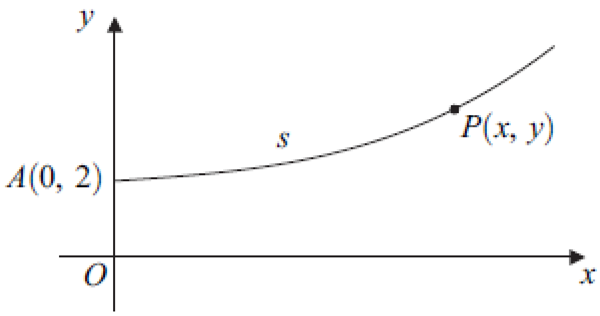
(i) $\frac{dy}{dx} = \frac{\sqrt{1-x^2}}{x}$; *(4 marks)*

(ii) the length of the arc of the curve from the point where $x = \frac{1}{4}$ to the point where $x = \frac{3}{4}$ is $\ln 3$. *(5 marks)*

**7** The diagram shows a curve which starts from the point $A$ with coordinates $(0, 2)$. The curve is such that, at every point $P$ on the curve,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}s$$

where $s$ is the length of the arc $AP$.




(a) (i) Show that

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{1}{2}\sqrt{4 + s^2}$$ *(3 marks)*

(ii) Hence show that

$$s = 2\sinh\frac{x}{2}$$ *(4 marks)*

(iii) Hence find the cartesian equation of the curve. *(3 marks)*

(b) Show that

$$y^2 = 4 + s^2$$ *(2 marks)*

January 2010

**5 (a)** Using the identities

$$\cosh^2 t - \sinh^2 t = 1\,, \quad \tanh t = \frac{\sinh t}{\cosh t} \quad \text{and} \quad \operatorname{sech} t = \frac{1}{\cosh t}$$

show that:

**(i)** $\tanh^2 t + \operatorname{sech}^2 t = 1\,;$ *(2 marks)*

**(ii)** $\dfrac{\mathrm{d}}{\mathrm{d}t}(\tanh t) = \operatorname{sech}^2 t\,;$ *(3 marks)*

**(iii)** $\dfrac{\mathrm{d}}{\mathrm{d}t}(\operatorname{sech} t) = -\operatorname{sech} t \tanh t\,.$ *(3 marks)*

**(b)** A curve $C$ is given parametrically by

$$x = \operatorname{sech} t\,, \quad y = 4 - \tanh t$$

**(i)** Show that the arc length, $s$, of $C$ between the points where $t = 0$ and $t = \frac{1}{2}\ln 3$ is given by

$$s = \int_0^{\frac{1}{2}\ln 3} \operatorname{sech} t \,\mathrm{d}t$$ *(4 marks)*

**(ii)** Using the substitution $u = \mathrm{e}^t$, find the exact value of $s$. *(6 marks)*

June 2010

**5 (a)** Using the identities

$$\cosh^2 t - \sinh^2 t = 1\,, \quad \tanh t = \frac{\sinh t}{\cosh t} \quad \text{and} \quad \operatorname{sech} t = \frac{1}{\cosh t}$$

show that:

**(i)** $\tanh^2 t + \operatorname{sech}^2 t = 1\,;$ *(2 marks)*

**(ii)** $\dfrac{\mathrm{d}}{\mathrm{d}t}(\tanh t) = \operatorname{sech}^2 t\,;$ *(3 marks)*

**(iii)** $\dfrac{\mathrm{d}}{\mathrm{d}t}(\operatorname{sech} t) = -\operatorname{sech} t \tanh t\,.$ *(3 marks)*

**(b)** A curve $C$ is given parametrically by

$$x = \operatorname{sech} t\,, \quad y = 4 - \tanh t$$

**(i)** Show that the arc length, $s$, of $C$ between the points where $t = 0$ and $t = \frac{1}{2}\ln 3$ is given by

$$s = \int_0^{\frac{1}{2}\ln 3} \operatorname{sech} t \,\mathrm{d}t$$ *(4 marks)*

**(ii)** Using the substitution $u = \mathrm{e}^t$, find the exact value of $s$. *(6 marks)*

January 2011

**6 (a)** Given that

$$x = \ln(\sec t + \tan t) - \sin t$$

show that

$$\frac{dx}{dt} = \sin t \tan t \qquad \textit{(4 marks)}$$

**(b)** A curve is given parametrically by the equations

$$x = \ln(\sec t + \tan t) - \sin t, \quad y = \cos t$$

The length of the arc of the curve between the points where $t = 0$ and $t = \frac{\pi}{3}$ is denoted by $s$.

Show that $s = \ln p$, where $p$ is an integer. *(6 marks)*

June 2011

**5 (a)** The arc of the curve $y^2 = x^2 + 8$ between the points where $x = 0$ and $x = 6$ is rotated through $2\pi$ radians about the $x$-axis. Show that the area $S$ of the curved surface formed is given by

$$S = 2\sqrt{2}\pi \int_0^6 \sqrt{x^2 + 4}\, dx \qquad \textit{(5 marks)}$$

**(b)** By means of the substitution $x = 2\sinh\theta$, show that

$$S = \pi(24\sqrt{5} + 4\sqrt{2}\sinh^{-1} 3) \qquad \textit{(8 marks)}$$

January 2012

**3** A curve has cartesian equation

$$y = \tfrac{1}{2}\ln(\tanh x)$$

**(a)** Show that

$$\frac{dy}{dx} = \frac{1}{\sinh 2x} \qquad \textit{(4 marks)}$$

**(b)** The points $A$ and $B$ on the curve have $x$-coordinates $\ln 2$ and $\ln 4$ respectively. Find the arc length $AB$, giving your answer in the form $p \ln q$, where $p$ and $q$ are rational numbers. *(8 marks)*

June 2012

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|---|---|---|---|
| **6 (a)** | Show that $$\frac{1}{4}(\cosh 4x + 2\cosh 2x + 1) = \cosh^2 x \cosh 2x$$ | *(3 marks)* |
| **(b)** | Show that, if $y = \cosh^2 x$, then $$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \cosh^2 2x$$ | *(3 marks)* |
| **(c)** | The arc of the curve $y = \cosh^2 x$ between the points where $x = 0$ and $x = \ln 2$ is rotated through $2\pi$ radians about the $x$-axis. Show that the area $S$ of the curved surface formed is given by $$S = \frac{\pi}{256}(a \ln 2 + b)$$ where $a$ and $b$ are integers. | *(7 marks)* |

January 2013

| | | |
|---|---|---|
| **6** | A curve is defined parametrically by $$x = t^3 + 5, \qquad y = 6t^2 - 1$$ The arc length between the points where $t = 0$ and $t = 3$ on the curve is $s$. | |
| **(a)** | Show that $s = \int_0^3 3t\sqrt{t^2 + A}\ \mathrm{d}t$, stating the value of the constant $A$. | *(4 marks)* |
| **(b)** | Hence show that $s = 61$. | *(4 marks)* |

**7 (a) (i)** Show that

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(2u\sqrt{1+4u^2}+\sinh^{-1}2u\right)=k\sqrt{1+4u^2}$$

where $k$ is an integer. *(4 marks)*

**(ii)** Hence show that

$$\int_0^1 \sqrt{1+4u^2}\ \mathrm{d}u = p\sqrt{5}+q\sinh^{-1}2$$

where $p$ and $q$ are rational numbers. *(2 marks)*

**(b)** The arc of the curve with equation $y=\frac{1}{2}\cos 4x$ between the points where $x=0$ and $x=\frac{\pi}{8}$ is rotated through $2\pi$ radians about the $x$-axis.

**(i)** Show that the area $S$ of the curved surface formed is given by

$$S=\pi\int_0^{\frac{\pi}{8}}\cos 4x\sqrt{1+4\sin^2 4x}\ \mathrm{d}x$$ *(2 marks)*

**(ii)** Use the substitution $u=\sin 4x$ to find the exact value of $S$. *(4 marks)*