FP2: Summations

Past Paper Questions
2006 - 2013

Name:

January 2006

1 (a) Show that

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2} \qquad \textit{(2 marks)}$$

(b) Hence find the sum of the first n terms of the series

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \ldots \qquad \textit{(4 marks)}$$

4 (a) Prove by induction that

$$2 + (3 \times 2) + (4 \times 2^2) + \ldots + (n+1)\,2^{n-1} = n\,2^n$$

for all integers $n \geqslant 1$. *(6 marks)*

(b) Show that

$$\sum_{r=n+1}^{2n} (r+1)\,2^{r-1} = n\,2^n(2^{n+1} - 1) \qquad \textit{(3 marks)}$$

June 2006

1 (a) Given that

$$\frac{r^2 + r - 1}{r(r+1)} = A + B\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

find the values of A and B. *(3 marks)*

(b) Hence find the value of

$$\sum_{r=1}^{99} \frac{r^2 + r - 1}{r(r+1)} \qquad \textit{(4 marks)}$$

6 (a) The function f is given by

$$\mathrm{f}(n) = 15^n - 8^{n-2}$$

Express

$$\mathrm{f}(n+1) - 8\mathrm{f}(n)$$

in the form $k \times 15^n$. *(4 marks)*

(b) Prove by induction that $15^n - 8^{n-2}$ is a multiple of 7 for all integers $n \geqslant 2$. *(4 marks)*

February 2007

7 (a) Use the identity $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$ with $A = (r+1)x$ and $B = rx$ to show that

$$\tan rx \tan(r+1)x = \frac{\tan(r+1)x}{\tan x} - \frac{\tan rx}{\tan x} - 1$$ *(4 marks)*

(b) Use the method of differences to show that

$$\tan\frac{\pi}{50}\tan\frac{2\pi}{50} + \tan\frac{2\pi}{50}\tan\frac{3\pi}{50} + \dots + \tan\frac{19\pi}{50}\tan\frac{20\pi}{50} = \frac{\tan\frac{2\pi}{5}}{\tan\frac{\pi}{50}} - 20$$ *(5 marks)*

June 2007

1 (a) Given that $\mathrm{f}(r) = (r-1)r^2$, show that

$$\mathrm{f}(r+1) - \mathrm{f}(r) = r(3r+1)$$ *(3 marks)*

(b) Use the method of differences to find the value of

$$\sum_{r-50}^{99} r(3r+1)$$ *(4 marks)*

6 (a) Show that

$$\left(1 - \frac{1}{(k+1)^2}\right) \times \frac{k+1}{2k} = \frac{k+2}{2(k+1)}$$ *(3 marks)*

(b) Prove by induction that for all integers $n \geqslant 2$

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\dots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$$ *(4 marks)*

January 2008

2 (a) Show that

$$(2r+1)^3 - (2r-1)^3 = 24r^2 + 2$$ *(3 marks)*

(b) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$ *(6 marks)*

5 Prove by induction that for all integers $n \geqslant 1$

$$\sum_{r-1}^{n} (r^2+1)(r!) = n(n+1)!$$ *(7 marks)*

June 2008

2 (a) Given that

$$\frac{1}{r(r+1)(r+2)} = \frac{A}{r(r+1)} + \frac{B}{(r+1)(r+2)}$$

show that $A = \frac{1}{2}$ and find the value of B. *(3 marks)*

(b) Use the method of differences to find

$$\sum_{r=10}^{98} \frac{1}{r(r+1)(r+2)}$$

giving your answer as a rational number. *(4 marks)*

7 (a) Explain why $n(n+1)$ is a multiple of 2 when n is an integer. *(1 mark)*

(b) (i) Given that

$$\mathrm{f}(n) = n(n^2 + 5)$$

show that $\mathrm{f}(k+1) - \mathrm{f}(k)$, where k is a positive integer, is a multiple of 6. *(4 marks)*

(ii) Prove by induction that $\mathrm{f}(n)$ is a multiple of 6 for all integers $n \geqslant 1$. *(4 marks)*

January 2009

3 (a) Given that $\mathrm{f}(r) = \frac{1}{4}r^2(r+1)^2$, show that

$$\mathrm{f}(r) - \mathrm{f}(r-1) = r^3$$

(3 marks)

(b) Use the method of differences to show that

$$\sum_{r=n}^{2n} r^3 = \frac{3}{4}n^2(n+1)(5n+1)$$

(5 marks)

6 Prove by induction that

$$\frac{2 \times 1}{2 \times 3} + \frac{2^2 \times 2}{3 \times 4} + \frac{2^3 \times 3}{4 \times 5} + \ldots + \frac{2^n \times n}{(n+1)(n+2)} = \frac{2^{n+1}}{n+2} - 1$$

for all integers $n \geqslant 1$. *(7 marks)*

June 2009

2 (a) Given that

$$\frac{1}{4r^2-1} = \frac{A}{2r-1} + \frac{B}{2r+1}$$

find the values of A and B. *(2 marks)*

(b) Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{n}{2n+1}$$ *(3 marks)*

(c) Find the least value of n for which $\sum_{r=1}^{n} \frac{1}{4r^2-1}$ differs from 0.5 by less than 0.001 .

(3 marks)

January 2010

5 The sum to r terms, S_r, of a series is given by

$$S_r = r^2(r+1)(r+2)$$

Given that u_r is the rth term of the series whose sum is S_r, show that:

(a) (i) $u_1 = 6$; *(1 mark)*

(ii) $u_2 = 42$; *(1 mark)*

(iii) $u_n = n(n+1)(4n-1)$. *(3 marks)*

(b) Show that

$$\sum_{r=n+1}^{2n} u_r = 3n^2(n+1)(5n+2)$$ *(3 marks)*

7 The sequence u_1, u_2, u_3,... is defined by

$$u_1 = 2 , \quad u_{k+1} = 2u_k + 1$$

(a) Prove by induction that, for all $n \geqslant 1$,

$$u_n = 3 \times 2^{n-1} - 1$$ *(5 marks)*

(b) Show that

$$\sum_{r=1}^{n} u_r = u_{n+1} - (n+2)$$ *(3 marks)*

June 2010

2 (a) Express $\frac{1}{r(r+2)}$ in partial fractions. *(3 marks)*

(b) Use the method of differences to find

$$\sum_{r=1}^{48} \frac{1}{r(r+2)}$$

giving your answer as a rational number. *(5 marks)*

6 (a) Show that $\frac{1}{(k+2)!} - \frac{k+1}{(k+3)!} = \frac{2}{(k+3)!}$. *(2 marks)*

(b) Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{r \times 2^r}{(r+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$$ *(6 marks)*

January 2011

2 (a) Given that

$$u_r = \frac{1}{6} r(r+1)(4r+11)$$

show that

$$u_r - u_{r-1} = r(2r+3)$$ *(3 marks)*

(b) Hence find the sum of the first hundred terms of the series

$$1 \times 5 + 2 \times 7 + 3 \times 9 + \ldots + r(2r+3) + \ldots$$ *(3 marks)*

7 (a) Given that

$$\mathrm{f}(k) = 12^k + 2 \times 5^{k-1}$$

show that

$$\mathrm{f}(k+1) - 5\mathrm{f}(k) = a \times 12^k$$

where a is an integer. *(3 marks)*

(b) Prove by induction that $12^n + 2 \times 5^{n-1}$ is divisible by 7 for all integers $n \geqslant 1$.
(4 marks)

June 2011

3 (a) Show that

$$(r+1)! - (r-1)! = (r^2+r-1)(r-1)!$$ *(2 marks)*

(b) Hence show that

$$\sum_{r=1}^{n}(r^2+r-1)(r-1)! = (n+2)n! - 2$$ *(4 marks)*

6 (a) Show that

$$(k+1)(4(k+1)^2-1) = 4k^3+12k^2+11k+3$$ *(2 marks)*

(b) Prove by induction that, for all integers $n \geqslant 1$,

$$1^2+3^2+5^2+\ldots+(2n-1)^2 = \frac{1}{3}n(4n^2-1)$$ *(6 marks)*

January 2012

4 The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = \frac{3}{4} \qquad u_{n+1} = \frac{3}{4-u_n}$$

Prove by induction that, for all $n \geqslant 1$,

$$u_n = \frac{3^{n+1}-3}{3^{n+1}-1}$$ *(6 marks)*

June 2012

3 (a) Show that

$$\frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} = \frac{r2^r}{(r+1)(r+2)}$$ *(3 marks)*

(b) Hence find

$$\sum_{r=1}^{30} \frac{r2^r}{(r+1)(r+2)}$$

giving your answer in the form $2^n - 1$, where n is an integer. *(3 marks)*

7 (a) Prove by induction that, for all integers $n \geqslant 1$,

$$\frac{3}{1^2\times2^2}+\frac{5}{2^2\times3^2}+\frac{7}{3^2\times4^2}+\ldots+\frac{2n+1}{n^2(n+1)^2} = 1-\frac{1}{(n+1)^2}$$ *(7 marks)*

(b) Find the smallest integer n for which the sum of the series differs from 1 by less than $10^{\ 5}$. *(2 marks)*

January 2013

3 (a) Show that $\frac{1}{5r-2} - \frac{1}{5r+3} = \frac{A}{(5r-2)(5r+3)}$, stating the value of the constant A. *(2 marks)*

(b) Hence use the method of differences to show that

$$\sum_{r-1}^{n} \frac{1}{(5r-2)(5r+3)} = \frac{n}{3(5n+3)}$$ *(4 marks)*

(c) Find the value of

$$\sum_{r-1}^{\infty} \frac{1}{(5r-2)(5r+3)}$$ *(1 mark)*

7 The polynomial $\mathrm{p}(n)$ is given by $\mathrm{p}(n) = (n-1)^3 + n^3 + (n+1)^3$.

(a) (i) Show that $\mathrm{p}(k+1) - \mathrm{p}(k)$, where k is a positive integer, is a multiple of 9. *(3 marks)*

(ii) Prove by induction that $\mathrm{p}(n)$ is a multiple of 9 for all integers $n \geqslant 1$. *(4 marks)*

(b) Using the result from part **(a)(ii)**, show that $n(n^2+2)$ is a multiple of 3 for any positive integer n. *(2 marks)*

June 2013

3 The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 2, \qquad u_{n+1} = \frac{5u_n - 3}{3u_n - 1}$$

Prove by induction that, for all integers $n \geqslant 1$,

$$u_n = \frac{3n+1}{3n-1}$$ *(6 marks)*

4 (a) Given that $\mathrm{f}(r) = r^2(2r^2 - 1)$, show that

$$\mathrm{f}(r) - \mathrm{f}(r-1) = (2r-1)^3$$ *(3 marks)*

(b) Use the method of differences to show that

$$\sum_{r=n+1}^{2n} (2r-1)^3 = 3n^2(10n^2 - 1)$$ *(4 marks)*