FP4: Applications of Vectors

Past Paper Questions
2006 - 2013

Name:

January 2006

3 (a) The plane Π has equation $\mathbf{r} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$.

(i) Find a vector which is perpendicular to both $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$. *(2 marks)*

(ii) Hence find an equation for Π in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(2 marks)*

(b) The line L has equation $\left(\mathbf{r} - \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}\right) \times \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \mathbf{0}$.

Verify that $\mathbf{r} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ is also an equation for L. *(2 marks)*

(c) Determine the position vector of the point of intersection of Π and L. *(4 marks)*

June 2006

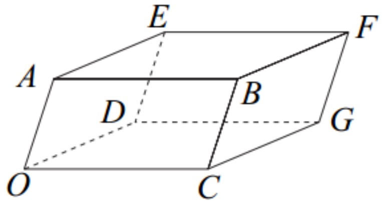
1 Two planes, Π_1 and Π_2, have equations $\mathbf{r} \cdot \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = 0$ and $\mathbf{r} \cdot \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = 0$ respectively.

(a) Determine the cosine of the acute angle between Π_1 and Π_2. *(4 marks)*

(b) (i) Find $\begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$. *(2 marks)*

(ii) Find a vector equation for the line of intersection of Π_1 and Π_2. *(2 marks)*

7 The diagram shows the parallelepiped $OABCDEFG$.



Points A, B, C and D have position vectors

$$\mathbf{a} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 6 \\ 1 \\ 6 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \text{ and } \mathbf{d} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

respectively, relative to the origin O.

(a) Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent. *(1 mark)*

(b) Determine the volume of the parallelepiped. *(3 marks)*

(c) Determine a vector equation for the plane $ABDG$:

(i) in the form $\mathbf{r} = \mathbf{u} + \lambda\mathbf{v} + \mu\mathbf{w}$; *(2 marks)*

(ii) in the form $\mathbf{r}\,.\,\mathbf{n} = d$. *(4 marks)*

(d) Find cartesian equations for the line OF, and hence find the direction cosines of this line. *(4 marks)*

January 2007

3 The points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively relative to an origin O, where

$$\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \ \mathbf{q} = \begin{bmatrix} -3 \\ 4 \\ 20 \end{bmatrix} \text{ and } \mathbf{r} = \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix}$$

(a) (i) Determine $\mathbf{p} \times \mathbf{q}$. *(2 marks)*

(ii) Find the area of triangle OPQ. *(3 marks)*

(b) Use the scalar triple product to show that $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are linearly dependent, and interpret this result geometrically. *(3 marks)*

5 (a) Find, to the nearest 0.1°, the acute angle between the planes with equations

$$\mathbf{r}\,.\,(3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) = 2 \quad \text{and} \quad \mathbf{r}\,.\,(2\mathbf{i} + 12\mathbf{j} - \mathbf{k}) = 38$$ *(4 marks)*

(b) Write down cartesian equations for these two planes. *(2 marks)*

(c) (i) Find, in the form $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$, cartesian equations for the line of intersection of the two planes. *(5 marks)*

(ii) Determine the direction cosines of this line. *(2 marks)*

June 2007

5 The line l has equation $\mathbf{r} = \begin{bmatrix} 3 \\ 26 \\ -15 \end{bmatrix} + \lambda \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix}$.

(a) Show that the point $P(-29, 42, -19)$ lies on l. *(1 mark)*

(b) Find:

(i) the direction cosines of l; *(2 marks)*

(ii) the acute angle between l and the z-axis. *(1 mark)*

(c) The plane Π has cartesian equation $3x - 4y + 5z = 100$.

(i) Write down a normal vector to Π. *(1 mark)*

(ii) Find the acute angle between l and this normal vector. *(4 marks)*

(d) Find the position vector of the point Q where l meets Π. *(4 marks)*

(e) Determine the shortest distance from P to Π. *(3 marks)*

January 2008

6 (a) The line l has equation $\mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$.

(i) Write down a vector equation for l in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. *(1 mark)*

(ii) Write down cartesian equations for l. *(2 marks)*

(iii) Find the direction cosines of l and explain, geometrically, what these represent. *(3 marks)*

(b) The plane Π has equation $\mathbf{r} = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

(i) Find an equation for Π in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(4 marks)*

(ii) State the geometrical significance of the value of d in this case. *(1 mark)*

(c) Determine, to the nearest 0.1°, the angle between l and Π. *(4 marks)*

June 2008

4 Two planes have equations

$$\mathbf{r} \cdot \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} = 12 \quad \text{and} \quad \mathbf{r} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 7$$

(a) Find, to the nearest 0.1°, the acute angle between the two planes. *(4 marks)*

(b) (i) The point $P(0, a, b)$ lies in both planes. Find the value of a and the value of b. *(3 marks)*

(ii) By using a vector product, or otherwise, find a vector which is parallel to both planes. *(2 marks)*

(iii) Find a vector equation for the line of intersection of the two planes. *(2 marks)*

January 2009

1 The line l has equation $\mathbf{r} = (1 + 4t)\mathbf{i} + (-2 + 12t)\mathbf{j} + (1 - 3t)\mathbf{k}$.

(a) Write down a direction vector for l. *(1 mark)*

(b) (i) Find direction cosines for l. *(2 marks)*

(ii) Explain the geometrical significance of the direction cosines in relation to l. *(1 mark)*

(c) Write down a vector equation for l in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. *(2 marks)*

6 The line L and the plane Π are, respectively, given by the equations

$$\mathbf{r} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{r} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 20$$

(a) Determine the size of the acute angle between L and Π. *(4 marks)*

(b) The point P has coordinates $(10, -5, 37)$.

(i) Show that P lies on L. *(1 mark)*

(ii) Find the coordinates of the point Q where L meets Π. *(4 marks)*

(iii) Deduce the distance PQ and the shortest distance from P to Π. *(3 marks)*

June 2009

3 The plane Π has equation $\mathbf{r} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$.

(a) Find an equation for Π in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(4 marks)*

(b) Show that the line with equation $\mathbf{r} = \begin{bmatrix} 7 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix}$ does not intersect Π, and explain the geometrical significance of this result. *(4 marks)*

5 The points A, B, C and D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively, relative to the origin O, where

$$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \text{ and } \mathbf{d} = \begin{bmatrix} 5 \\ 5 \\ 11 \end{bmatrix}$$

(a) Using scalar triple products:

(i) show that $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are coplanar; *(2 marks)*

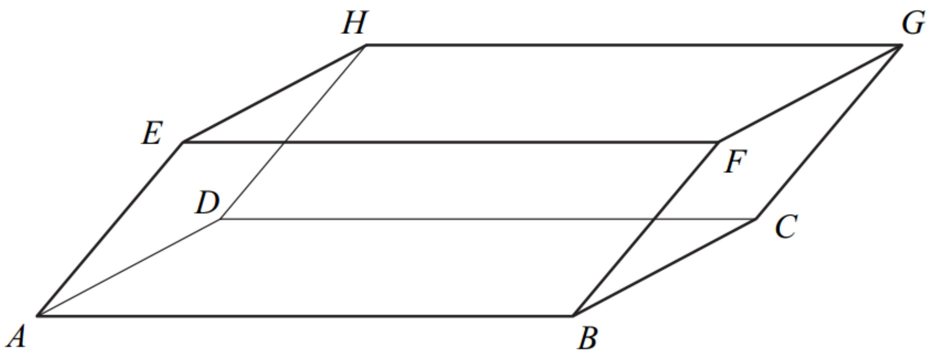
(ii) find the volume of the parallelepiped defined by AB, AC and AD. *(4 marks)*

(b) (i) Find the direction ratios of the line BD. *(2 marks)*

(ii) Deduce the direction cosines of the line BD. *(2 marks)*

January 2010

2 The diagram shows the parallelepiped $ABCDEFGH$.



The position vectors of A, B, C, D and E are, respectively,

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -3 \\ 10 \\ 4 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} -7 \\ 10 \\ 7 \end{bmatrix} \quad \text{and} \quad \mathbf{e} = \begin{bmatrix} 3 \\ 4 \\ 10 \end{bmatrix}$$

(a) Show that the area of $ABCD$ is 37. *(4 marks)*

(b) Find the volume of $ABCDEFGH$. *(2 marks)*

(c) Deduce the distance between the planes $ABCD$ and $EFGH$. *(2 marks)*

6 (a) Find the value of p for which the planes with equations

$$\mathbf{r} \cdot \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix} = 42 \quad \text{and} \quad \mathbf{r} \cdot \begin{bmatrix} 4p+1 \\ p-2 \\ 1 \end{bmatrix} = -7$$

(i) are perpendicular; *(3 marks)*

(ii) are parallel. *(3 marks)*

(b) In the case when $p = 4$:

(i) write down a cartesian equation for each plane; *(2 marks)*

(ii) find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, an equation for l, the line of intersection of the planes. *(6 marks)*

(c) Determine a vector equation, in the form $\mathbf{r} = \mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$, for the plane which contains l and which passes through the point $(30, 7, 30)$. *(2 marks)*

June 2010

3 The plane Π_1 is perpendicular to the vector $9\mathbf{i} - 8\mathbf{j} + 72\mathbf{k}$ and passes through the point $A(2, 10, 1)$.

(a) Find, in the form $\mathbf{r}.\mathbf{n} = d$, a vector equation for Π_1. *(3 marks)*

(b) Determine the exact value of the cosine of the acute angle between Π_1 and the plane Π_2 with equation $\mathbf{r}.(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 11$. *(4 marks)*

4 The fixed points A and B and the variable point C have position vectors

$$\mathbf{a} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2-t \\ t \\ 5 \end{bmatrix}$$

respectively, relative to the origin O, where t is a scalar parameter.

(a) Find an equation of the line AB in the form $(\mathbf{r} - \mathbf{u}) \times \mathbf{v} = \mathbf{0}$. *(3 marks)*

(b) Determine $\mathbf{b} \times \mathbf{c}$ in terms of t. *(4 marks)*

(c) (i) Show that $\mathbf{a} . (\mathbf{b} \times \mathbf{c})$ is constant for all values of t, and state the value of this constant. *(2 marks)*

(ii) Write down a geometrical conclusion that can be deduced from the answer to part **(c)(i)**. *(1 mark)*

6 The line L and the plane Π have vector equations

$$\mathbf{r} = \begin{bmatrix} 7 \\ 8 \\ 50 \end{bmatrix} + t \begin{bmatrix} 6 \\ 2 \\ -9 \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} -2 \\ 0 \\ -25 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$$

respectively.

(a) (i) Find direction cosines for L. *(2 marks)*

(ii) Show that L is perpendicular to Π. *(3 marks)*

(b) For the system of equations

$$\begin{aligned} 6p + 5q + r &= 9 \\ 2p + 3q + 6r &= 8 \\ -9p + 4q + 2r &= 75 \end{aligned}$$

form a pair of equations in p and q only, and hence find the unique solution of this system of equations. *(5 marks)*

(c) It is given that L meets Π at the point P.

(i) Demonstrate how the coordinates of P may be obtained from the system of equations in part **(b)**. *(2 marks)*

(ii) Hence determine the coordinates of P. *(2 marks)*

5 The planes Π_1 and Π_2 have vector equations $\mathbf{r}\,.\begin{bmatrix}6\\2\\9\end{bmatrix} = 5$ and $\mathbf{r}\,.\begin{bmatrix}10\\-1\\-11\end{bmatrix} = 4$ respectively.

(a) Write down cartesian equations for Π_1 and Π_2. *(1 mark)*

(b) Find a vector equation for the line of intersection of Π_1 and Π_2. *(5 marks)*

(c) The plane Π_3 has cartesian equation $5x + 3y + 11z = 28$.

Use your answer to part **(b)** to find the coordinates of the point of intersection of Π_1, Π_2 and Π_3. *(4 marks)*

(d) Determine a vector equation for the plane which passes through the point $(4, 1, 9)$ and which is perpendicular to both Π_1 and Π_2. *(3 marks)*

6 The plane Π has equation $\mathbf{r}\,.\begin{bmatrix}12\\15\\16\end{bmatrix} = 11$ and the point Q has coordinates $(1, 1, -1)$.

(a) Show that Q is in Π. *(1 mark)*

(b) (i) Write down cartesian equations for the line l which passes through Q and is perpendicular to Π. *(2 marks)*

(ii) Deduce the direction cosines of l. *(2 marks)*

(c) The points M and N are on l, and each is 50 units from Π.

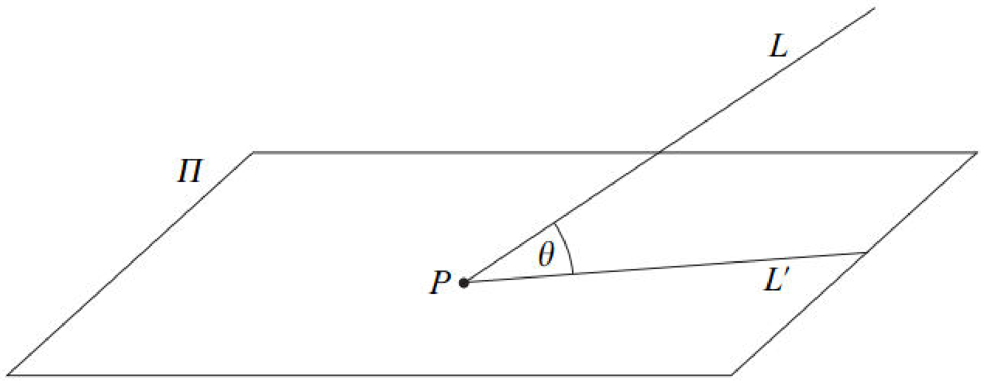
Find the coordinates of M and N. *(3 marks)*

(d) Given that the point $P(5, 1, -4)$ is in Π, determine the area of triangle PMN. *(3 marks)*

8 The diagram shows the plane Π and the lines L and L'. The plane Π and the line L have equations

$$\mathbf{r} \cdot \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix} = 37 \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

The line L does not lie in Π, and intersects it at the point P.



(a) Determine the value of θ, the angle between L and Π, giving your answer to the nearest 0.1°. *(4 marks)*

(b) Find the coordinates of P. *(4 marks)*

(c) The line L' lies in Π and is such that the angle between L and L' is θ, the angle between L and Π.

(i) Find a vector which is parallel to Π and perpendicular to L. *(3 marks)*

(ii) Hence, or otherwise, find a vector equation for L' in the form $\mathbf{r} = \mathbf{a} + \mu\mathbf{b}$. *(4 marks)*

January 2012

6 The planes Π_1 and Π_2 have equations

$$\mathbf{r} \,.\, \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} = 10 \quad \text{and} \quad \mathbf{r} \,.\, \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = 7$$

respectively.

(a) Determine, to the nearest degree, the acute angle between Π_1 and Π_2. *(4 marks)*

(b) By setting $z = t$, find cartesian equations for the line of intersection of Π_1 and Π_2 in the form

$$\frac{x-a}{l} = \frac{y-b}{m} = z = t$$ *(6 marks)*

(c) The line L, with equation $\mathbf{r} = \begin{bmatrix} 20 \\ -1 \\ 7 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 9 \\ 4 \end{bmatrix}$, intersects Π_1 at the point P and Π_2 at the point Q.

Show that $PQ = k\sqrt{2}$, where k is an integer. *(6 marks)*

June 2012

2 A line has vector equation $\left(\mathbf{r} - \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}\right) \times \begin{bmatrix} 4 \\ 7 \\ -4 \end{bmatrix} = \mathbf{0}$.

(a) Determine the direction cosines of this line. *(3 marks)*

(b) Explain the geometrical significance of the direction cosines in relation to the line. *(1 mark)*

4 The lines L_1 and L_2 have equations

$$\mathbf{r} = \begin{bmatrix} 7 \\ -25 \\ 9 \end{bmatrix} + \alpha \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} 7 \\ 19 \\ -2 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

respectively.

(a) Determine a vector, $\mathbf{n}$, which is perpendicular to both lines. *(2 marks)*

(b) (i) The point A on L_1 and the point B on L_2 are such that $\overrightarrow{AB} = \lambda\mathbf{n}$ for some constant λ.

Show that

$$3\alpha - 2\beta + 2\lambda = 0$$
$$4\alpha - 2\beta - 5\lambda = -44$$
$$7\alpha - 3\beta + 2\lambda = -11$$

(3 marks)

(ii) Find the position vectors of A and B. *(3 marks)*

(iii) Deduce the shortest distance between L_1 and L_2. *(2 marks)*

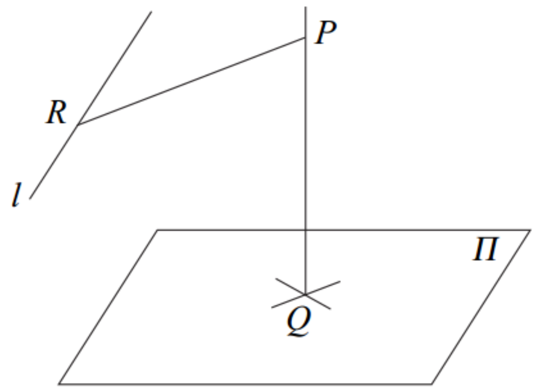
8 The point Q has position vector $\mathbf{q} = \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix}$, the plane Π has equation $\mathbf{r} . \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 36$,

and the line l has equation $\mathbf{r} = \begin{bmatrix} 20 \\ -8 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -7 \\ 5 \\ 3 \end{bmatrix}$.

(a) Show that Q lies in Π. *(1 mark)*

(b) Show also that l is parallel to Π. *(2 marks)*

(c) The diagram shows the point P, which lies on the normal to Π that passes through Q. The point R is the point on l which is closest to P, and $PQ = PR$.



Determine the coordinates of P. *(9 marks)*

January 2013

8 The four vertices of a parallelogram $ABCD$ have coordinates

$$A(1,\ 0,\ 2),\ B(3,\ -1,\ 5),\ C(7,\ 2,\ 4) \text{ and } D(5,\ 3,\ 1)$$

(a) (i) Find $\overrightarrow{AB} \times \overrightarrow{AD}$. *(3 marks)*

(ii) Show that the area of the parallelogram is $p\sqrt{10}$, where p is an integer to be found. *(2 marks)*

(b) The diagonals AC and BD of the parallelogram meet at the point M. The line L passes through M and is perpendicular to the plane $ABCD$.

Find an equation for the line L, giving your answer in the form $(\mathbf{r} - \mathbf{u}) \times \mathbf{v} = \mathbf{0}$. *(4 marks)*

(c) The plane Π is parallel to the plane $ABCD$ and passes through the point $Q(6,\ 5,\ 17)$.

(i) Find the coordinates of the point of intersection of the line L with the plane Π. *(6 marks)*

(ii) One face of a parallelepiped is $ABCD$ and the opposite face lies in the plane Π.

Find the volume of the parallelepiped. *(3 marks)*

June 2013

1 The points A, B, C and D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively relative to the origin O, where

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix},\quad \mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix},\quad \mathbf{c} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{d} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$$

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. *(3 marks)*

(b) The points A, B and C lie in the plane Π. Find a Cartesian equation for Π. *(2 marks)*

(c) Find the volume of the parallelepiped defined by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$. *(3 marks)*

8 A line and a plane have equations

$$\frac{x-3}{p} = \frac{y-q}{3} = \frac{z-1}{-1}$$

and

$$\mathbf{r} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 10$$

respectively, where p and q are constants.

(a) Show that the line is **not** perpendicular to the plane. *(1 mark)*

(b) In the case where the line lies in the plane, find the values of p and q. *(4 marks)*

(c) In the case where the angle, θ, between the line and the plane satisfies $\sin\theta = \frac{1}{\sqrt{6}}$, and the line intersects the plane at $z = 2$:

(i) find the value of p; *(5 marks)*

(ii) find the value of q. *(2 marks)*