FP4: Eigenvectors

Past Paper Questions
2006 - 2013

Name:

January 2006

7 The matrix $\mathbf{M} = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 1 \\ 3 & -5 & 3 \end{bmatrix}$.

(a) Given that $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are eigenvectors of **M**, find the eigenvalues corresponding to **u** and **v**. *(5 marks)*

(b) Given also that the third eigenvalue of **M** is 1, find a corresponding eigenvector. *(6 marks)*

(c) (i) Express the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in terms of **u** and **v**. *(1 mark)*

(ii) Deduce that $\mathbf{M}^n \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \lambda^n \mathbf{u} + \mu^n \mathbf{v}$, where λ and μ are scalar constants whose values should be stated. *(4 marks)*

(iii) Hence prove that, for all positive **odd** integers n,

$$\mathbf{M}^n \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2^n \\ 0 \\ 2^n \end{bmatrix}$$ *(3 marks)*

June 2006

8 For real numbers a and b, with $b \neq 0$ and $b \neq \pm a$, the matrix

$$\mathbf{M} = \begin{bmatrix} a & b+a \\ b-a & -a \end{bmatrix}$$

(a) (i) Show that the eigenvalues of **M** are b and $-b$. *(3 marks)*

(ii) Show that $\begin{bmatrix} b+a \\ b-a \end{bmatrix}$ is an eigenvector of **M** with eigenvalue b. *(2 marks)*

(iii) Find an eigenvector of **M** corresponding to the eigenvalue $-b$. *(2 marks)*

(b) By writing **M** in the form $\mathbf{UDU}^{-1}$, for some suitably chosen diagonal matrix **D** and corresponding matrix **U**, show that

$$\mathbf{M}^{11} = b^{10}\mathbf{M}$$ *(7 marks)*

January 2007

6 (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$

(6 marks)

(b) (i) Write down a diagonal matrix $\mathbf{D}$, and a suitable matrix $\mathbf{U}$, such that

$$\mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$$

(2 marks)

(ii) Write down also the matrix $\mathbf{U}^{-1}$. *(1 mark)*

(iii) Use your results from parts (b)(i) and (b)(ii) to determine the matrix $\mathbf{X}^5$ in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c and d are integers. *(3 marks)*

June 2007

7 (a) The matrix $\mathbf{M} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$ represents a shear.

(i) Find $\det \mathbf{M}$ and give a geometrical interpretation of this result. *(2 marks)*

(ii) Show that the characteristic equation of $\mathbf{M}$ is $\lambda^2 - 2\lambda + 1 = 0$, where λ is an eigenvalue of $\mathbf{M}$. *(2 marks)*

(iii) Hence find an eigenvector of $\mathbf{M}$. *(3 marks)*

(iv) Write down the equation of the line of invariant points of the shear. *(1 mark)*

(b) The matrix $\mathbf{S} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents a shear.

(i) Write down the characteristic equation of $\mathbf{S}$, giving the coefficients in terms of a, b, c and d. *(2 marks)*

(ii) State the numerical value of $\det \mathbf{S}$ and hence write down an equation relating a, b, c and d. *(2 marks)*

(iii) Given that the only eigenvalue of $\mathbf{S}$ is 1, find the value of $a + d$. *(2 marks)*

January 2008

4 The matrix $\mathbf{T}$ has eigenvalues 2 and -2, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ respectively.

(a) Given that $\mathbf{T} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$, where $\mathbf{D}$ is a diagonal matrix, write down suitable matrices $\mathbf{U}$, $\mathbf{D}$ and $\mathbf{U}^{-1}$. *(3 marks)*

(b) Hence prove that, for all **even** positive integers n,

$$\mathbf{T}^n = \mathrm{f}(n)\,\mathbf{I}$$

where $\mathrm{f}(n)$ is a function of n, and $\mathbf{I}$ is the 2×2 identity matrix. *(5 marks)*

7 The non-singular matrix $\mathbf{M} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

(a) (i) Show that

$$\mathbf{M}^2 + 2\mathbf{I} = k\mathbf{M}$$

for some integer k to be determined. *(3 marks)*

(ii) By multiplying the equation in part (a)(i) by $\mathbf{M}^{-1}$, show that

$$\mathbf{M}^{-1} = a\mathbf{M} + b\mathbf{I}$$

for constants a and b to be found. *(3 marks)*

(b) (i) Determine the characteristic equation of $\mathbf{M}$ and show that $\mathbf{M}$ has a repeated eigenvalue, 1, and another eigenvalue, 2. *(6 marks)*

(ii) Give a full set of eigenvectors for each of these eigenvalues. *(5 marks)*

(iii) State the geometrical significance of each set of eigenvectors in relation to the transformation with matrix $\mathbf{M}$. *(3 marks)*

June 2008

1 Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 7 & 12 \\ 12 & 0 \end{bmatrix}$. *(6 marks)*

7 A transformation T of three-dimensional space is given by the matrix $\mathbf{W} = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$.

(a) (i) Evaluate $\det \mathbf{W}$, and describe the geometrical significance of the answer in relation to T. *(2 marks)*

(ii) Determine the eigenvalues of $\mathbf{W}$. *(6 marks)*

(b) The plane H has equation $\mathbf{r} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0$.

(i) Write down a cartesian equation for H. *(1 mark)*

(ii) The point P has coordinates (a, b, c). Show that, whatever the values of a, b and c, the image of P under T lies in H. *(4 marks)*

January 2009

4 (a) Given that -1 is an eigenvalue of the matrix $\mathbf{M} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$, find a corresponding eigenvector. *(3 marks)*

(b) Determine the other two eigenvalues of $\mathbf{M}$, expressing each answer in its simplest surd form. *(8 marks)*

June 2009

6 The plane transformation T is defined by

$$\mathrm{T} : \begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $\mathbf{M} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$.

(a) Evaluate $\det \mathbf{M}$ and state the significance of this answer in relation to T. *(2 marks)*

(b) Find the single eigenvalue of $\mathbf{M}$ and a corresponding eigenvector. Describe the geometrical significance of these answers in relation to T. *(5 marks)*

(c) Show that the image of the line $y = \frac{1}{2}x + k$ under T is $y' = \frac{1}{2}x' + k$. *(3 marks)*

(d) Given that T is a shear, give a full geometrical description of this transformation. *(2 marks)*

7 The 2×2 matrix $\mathbf{M}$ has an eigenvalue 3, with corresponding eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and a second eigenvalue -3, with corresponding eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

The diagonalised form of $\mathbf{M}$ is $\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$.

(a) (i) Write down suitable matrices $\mathbf{D}$ and $\mathbf{U}$, and find $\mathbf{U}^{-1}$. *(4 marks)*

(ii) Hence determine the matrix $\mathbf{M}$. *(3 marks)*

(b) Given that n is a positive integer, use the result $\mathbf{M}^n = \mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$ to show that:

(i) when n is even, $\mathbf{M}^n = 3^n\,\mathbf{I}$;

(ii) when n is odd, $\mathbf{M}^n = 3^{n-1}\,\mathbf{M}$. *(6 marks)*

January 2010

7 (a) It is given that $\Delta = \begin{vmatrix} 16-q & 5 & 7 \\ -12 & -1-q & -7 \\ 6 & 6 & 10-q \end{vmatrix}$.

(i) By using row operations on the first two rows of Δ, show that $(4-q)$ is a factor of Δ. *(2 marks)*

(ii) Express Δ as the product of three linear factors. *(4 marks)*

(b) It is given that $\mathbf{M} = \begin{bmatrix} 16 & 5 & 7 \\ -12 & -1 & -7 \\ 6 & 6 & 10 \end{bmatrix}$.

(i) Verify that $\begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix}$ is an eigenvector of $\mathbf{M}$ and state its corresponding eigenvalue. *(3 marks)*

(ii) For each of the other two eigenvalues of $\mathbf{M}$, find a corresponding eigenvector. *(7 marks)*

(c) The transformation T has matrix $\mathbf{M}$. Write down cartesian equations for any one of the invariant lines of T. *(2 marks)*

June 2010

7 The transformation T is represented by the matrix $\mathbf{M}$ with diagonalised form

$$\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$$

where $\mathbf{U} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 27 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) (i) State the eigenvalues, and corresponding eigenvectors, of $\mathbf{M}$. *(4 marks)*

(ii) Find a cartesian equation for the line of invariant points of T. *(2 marks)*

(b) Write down $\mathbf{U}^{-1}$, and hence find the matrix $\mathbf{M}$ in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c and d are integers. *(5 marks)*

(c) By finding the element in the first row, first column position of $\mathbf{M}^n$, prove that

$$4 \times 3^{3n+1} + 1$$

is a multiple of 13 for all positive integers n. *(5 marks)*

January 2011

7 Let $\mathbf{Y} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

(a) Show that 4 is a repeated eigenvalue of $\mathbf{Y}$, and find the other eigenvalue of $\mathbf{Y}$. *(7 marks)*

(b) For each eigenvalue of $\mathbf{Y}$, find a full set of eigenvectors. *(5 marks)*

(c) The matrix $\mathbf{Y}$ represents the transformation T.

Describe the geometrical significance of the eigenvectors of $\mathbf{Y}$ in relation to T. *(3 marks)*

June 2011

5 (a) (i) Find the eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & 8 \end{bmatrix}$. *(6 marks)*

(ii) Hence write down each of the matrices $\mathbf{U}$, $\mathbf{D}$ and $\mathbf{U}^{-1}$ such that $\mathbf{A} = \mathbf{UDU}^{-1}$, where $\mathbf{D}$ is a diagonal matrix. *(4 marks)*

(b) A 2×2 matrix $\mathbf{M}$ has distinct real eigenvalues λ and μ, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(i) By considering the diagonalised form of $\mathbf{M}$, determine the eigenvalues of $\mathbf{M}^3$. *(2 marks)*

(ii) Write down the eigenvectors of $\mathbf{M}^3$. *(1 mark)*

January 2012

3 (a) Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{M} = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$. *(6 marks)*

(b) The plane transformation T is given by the matrix $\mathbf{M}$. Write down the coordinates of the invariant point of T. *(1 mark)*

June 2012

5 The matrix $\mathbf{M} = \begin{bmatrix} -11 & 9 \\ -16 & 13 \end{bmatrix}$ represents the plane transformation T.

(a) (i) Determine the eigenvalue, and a corresponding eigenvector, of $\mathbf{M}$. *(4 marks)*

(ii) Hence write down the value of m for which $y = mx$ is the invariant line of T which passes through the origin, and explain why it is actually a line of invariant points. *(2 marks)*

(iii) Show that, for this value of m, all lines with equations $y = mx + c$ are invariant lines of T. *(3 marks)*

(b) Given that T is a shear, give a full geometrical description of this transformation. *(2 marks)*

(c) Give a full geometrical description of the plane transformation represented by the matrix $\mathbf{M}^{-1}$. *(2 marks)*

January 2013

7 The matrix $\mathbf{M}$ is defined by

$$\mathbf{M} = \begin{bmatrix} -a & 0 & a \\ 0 & 6 & 0 \\ a & 0 & 2 \end{bmatrix}$$

where a is a real number. The distinct eigenvalues of $\mathbf{M}$ are λ_1, λ_2 and λ_3 with corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

(a) Given that $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, find λ_1. *(2 marks)*

(b) Given that $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, find the value of a. *(3 marks)*

(c) Given that $\lambda_3 = -6$, find a possible eigenvector $\mathbf{v}_3$. *(3 marks)*

(d) The matrix $\mathbf{M}$ can be expressed as $\mathbf{UDU}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

Write down possible matrices $\mathbf{D}$ and $\mathbf{U}$. *(3 marks)*

June 2013

5 The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix}$$

(a) Show that $\lambda = 2$ is an eigenvalue for $\mathbf{M}$, and find the other two eigenvalues. *(5 marks)*

(b) Find an eigenvector that corresponds to $\lambda = 2$. *(3 marks)*

(c) The matrix $\mathbf{N}$ is given by

$$\mathbf{N} = \begin{bmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{bmatrix}$$

(i) Show that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector for $\mathbf{N}$, and find the corresponding eigenvalue. *(2 marks)*

(ii) Hence state one eigenvector for the matrix $\mathbf{MN}$, and find the corresponding eigenvalue. *(3 marks)*