FP4:
Greatest Hits

Past Paper Questions
2006 - 2013

Name:

January 2006

7 The matrix $\mathbf{M} = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 1 \\ 3 & -5 & 3 \end{bmatrix}$.

(a) Given that $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are eigenvectors of **M**, find the eigenvalues corresponding to **u** and **v**. *(5 marks)*

(b) Given also that the third eigenvalue of **M** is 1, find a corresponding eigenvector. *(6 marks)*

(c) (i) Express the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in terms of **u** and **v**. *(1 mark)*

(ii) Deduce that $\mathbf{M}^n \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \lambda^n \mathbf{u} + \mu^n \mathbf{v}$, where λ and μ are scalar constants whose values should be stated. *(4 marks)*

(iii) Hence prove that, for all positive **odd** integers n,

$$\mathbf{M}^n \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2^n \\ 0 \\ 2^n \end{bmatrix}$$ *(3 marks)*

June 2006

8 For real numbers a and b, with $b \neq 0$ and $b \neq \pm a$, the matrix

$$\mathbf{M} = \begin{bmatrix} a & b+a \\ b-a & -a \end{bmatrix}$$

(a) (i) Show that the eigenvalues of **M** are b and $-b$. *(3 marks)*

(ii) Show that $\begin{bmatrix} b+a \\ b-a \end{bmatrix}$ is an eigenvector of **M** with eigenvalue b. *(2 marks)*

(iii) Find an eigenvector of **M** corresponding to the eigenvalue $-b$. *(2 marks)*

(b) By writing **M** in the form $\mathbf{UDU}^{-1}$, for some suitably chosen diagonal matrix **D** and corresponding matrix **U**, show that

$$\mathbf{M}^{11} = b^{10}\mathbf{M}$$ *(7 marks)*

January 2007

7 The transformation S is a shear with matrix $\mathbf{M} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$. Points (x, y) are mapped under S to image points (x', y') such that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

(a) Find the equation of the line of invariant points of S. *(2 marks)*

(b) Show that all lines of the form $y = x + c$, where c is a constant, are invariant lines of S. *(3 marks)*

(c) Evaluate $\det \mathbf{M}$, and state the property of shears which is indicated by this result. *(2 marks)*

(d) Calculate, to the nearest degree, the acute angle between the line $y = -x$ and its image under S. *(3 marks)*

June 2007

5 The line l has equation $\mathbf{r} = \begin{bmatrix} 3 \\ 26 \\ -15 \end{bmatrix} + \lambda \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix}$.

(a) Show that the point $P(-29, 42, -19)$ lies on l. *(1 mark)*

(b) Find:

(i) the direction cosines of l; *(2 marks)*

(ii) the acute angle between l and the z-axis. *(1 mark)*

(c) The plane Π has cartesian equation $3x - 4y + 5z = 100$.

(i) Write down a normal vector to Π. *(1 mark)*

(ii) Find the acute angle between l and this normal vector. *(4 marks)*

(d) Find the position vector of the point Q where l meets Π. *(4 marks)*

(e) Determine the shortest distance from P to Π. *(3 marks)*

January 2008

7 The non-singular matrix $\mathbf{M} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

(a) (i) Show that

$$\mathbf{M}^2 + 2\mathbf{I} = k\mathbf{M}$$

for some integer k to be determined. *(3 marks)*

(ii) By multiplying the equation in part (a)(i) by $\mathbf{M}^{-1}$, show that

$$\mathbf{M}^{-1} = a\mathbf{M} + b\mathbf{I}$$

for constants a and b to be found. *(3 marks)*

(b) (i) Determine the characteristic equation of **M** and show that **M** has a repeated eigenvalue, 1, and another eigenvalue, 2. *(6 marks)*

(ii) Give a full set of eigenvectors for each of these eigenvalues. *(5 marks)*

(iii) State the geometrical significance of each set of eigenvectors in relation to the transformation with matrix **M**. *(3 marks)*

June 2008

7 A transformation T of three-dimensional space is given by the matrix $\mathbf{W} = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$.

(a) (i) Evaluate det **W**, and describe the geometrical significance of the answer in relation to T. *(2 marks)*

(ii) Determine the eigenvalues of **W**. *(6 marks)*

(b) The plane H has equation $\mathbf{r} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0$.

(i) Write down a cartesian equation for H. *(1 mark)*

(ii) The point P has coordinates (a, b, c). Show that, whatever the values of a, b and c, the image of P under T lies in H. *(4 marks)*

January 2009

8 The plane transformation T has matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, and maps points (x, y) onto image points (X, Y) such that

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

(a) (i) Find $\mathbf{A}^{-1}$. *(2 marks)*

(ii) Hence express each of x and y in terms of X and Y. *(2 marks)*

(b) Give a full geometrical description of T. *(5 marks)*

(c) Any plane curve with equation of the form $\frac{x^2}{p} + \frac{y^2}{q} = 1$, where p and q are distinct positive constants, is an ellipse.

(i) Show that the curve E with equation $6x^2 + y^2 = 3$ is an ellipse. *(1 mark)*

(ii) Deduce that the image of the curve E under T has equation

$$2X^2 + 4XY + 5Y^2 = 15$$ *(2 marks)*

(iii) Explain why the curve with equation $2x^2 + 4xy + 5y^2 = 15$ is an ellipse. *(1 mark)*

June 2009

7 The 2×2 matrix $\mathbf{M}$ has an eigenvalue 3, with corresponding eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and a second eigenvalue -3, with corresponding eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

The diagonalised form of $\mathbf{M}$ is $\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$.

(a) (i) Write down suitable matrices $\mathbf{D}$ and $\mathbf{U}$, and find $\mathbf{U}^{-1}$. *(4 marks)*

(ii) Hence determine the matrix $\mathbf{M}$. *(3 marks)*

(b) Given that n is a positive integer, use the result $\mathbf{M}^n = \mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$ to show that:

(i) when n is even, $\mathbf{M}^n = 3^n\,\mathbf{I}$;

(ii) when n is odd, $\mathbf{M}^n = 3^{n-1}\,\mathbf{M}$. *(6 marks)*

January 2010

7 (a) It is given that $\Delta = \begin{vmatrix} 16-q & 5 & 7 \\ -12 & -1-q & -7 \\ 6 & 6 & 10-q \end{vmatrix}$.

(i) By using row operations on the first two rows of Δ, show that $(4-q)$ is a factor of Δ. *(2 marks)*

(ii) Express Δ as the product of three linear factors. *(4 marks)*

(b) It is given that $\mathbf{M} = \begin{bmatrix} 16 & 5 & 7 \\ -12 & -1 & -7 \\ 6 & 6 & 10 \end{bmatrix}$.

(i) Verify that $\begin{bmatrix} 2 \\ 5 \\ -7 \end{bmatrix}$ is an eigenvector of $\mathbf{M}$ and state its corresponding eigenvalue. *(3 marks)*

(ii) For each of the other two eigenvalues of $\mathbf{M}$, find a corresponding eigenvector. *(7 marks)*

(c) The transformation T has matrix $\mathbf{M}$. Write down cartesian equations for any one of the invariant lines of T. *(2 marks)*

June 2010

8 The matrix $\begin{bmatrix} 12 & 16 \\ -9 & 36 \end{bmatrix}$ represents the transformation which is the composition, in either order, of the two plane transformations

E: an enlargement, centre O and scale factor k $(k > 0)$

and

S: a shear parallel to the line l which passes through O

Show that $k = 24$ and find a cartesian equation for l. *(7 marks)*

January 2011

8 The plane transformation T is represented by the matrix $\mathbf{M} = \begin{bmatrix} -3 & 8 \\ -1 & 3 \end{bmatrix}$.

(a) The quadrilateral $ABCD$ has image $A'B'C'D'$ under T.

Evaluate $\det \mathbf{M}$ and describe the geometrical significance of both its sign and its magnitude in relation to $ABCD$ and $A'B'C'D'$. *(3 marks)*

(b) The line $y = px$ is a line of invariant points of T, and the line $y = qx$ is an invariant line of T.

Show that $p = \frac{1}{2}$ and determine the value of q. *(5 marks)*

(c) (i) Find the 2×2 matrix $\mathbf{R}$ which represents a reflection in the line $y = \frac{1}{2}x$. *(2 marks)*

(ii) Given that T is the composition of a shear, with matrix $\mathbf{S}$, followed by a reflection in the line $y = \frac{1}{2}x$, determine the matrix $\mathbf{S}$ and describe the shear as fully as possible. *(5 marks)*

June 2011

6 (a) The transformation U of three-dimensional space is represented by the matrix

$$\begin{bmatrix} 1 & 4 & -3 \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

(i) Write down a vector equation for the line L with cartesian equation

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$$ *(2 marks)*

(ii) Find a vector equation for the image of L under U, and deduce that it is a line through the origin. *(4 marks)*

(b) The plane transformation V is represented by the matrix $\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$.

L_1 is the line with equation $y = \frac{1}{2}x + k$, and L_2 is the image of L_1 under V.

(i) Find, in the form $y = mx + c$, the cartesian equation for L_2. *(4 marks)*

(ii) Deduce that L_2 is parallel to L_1 and find, in terms of k, the distance between these two lines. *(3 marks)*

7 The plane transformation T is a rotation through θ radians anticlockwise about O, and maps points (x, y) onto image points (X, Y) such that

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $c = \cos\theta$ and $s = \sin\theta$.

(a) Write down the inverse of the matrix $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ and hence show that

$$x = cX + sY \quad \text{and} \quad y = -sX + cY$$

(3 marks)

(b) The curve C has equation $x^2 - 6xy - 7y^2 = 8$.

The image of C under T is the curve C' with equation $pX^2 + qXY + rY^2 = 8$.

(i) Use the results of part **(a)** to show that

$$q = 6s^2 + 16sc - 6c^2$$

and express p and r similarly in terms of c and s. *(4 marks)*

(ii) Given that θ is an acute angle, find the values of c and s for which $q = 0$ and hence in this case express the equation of C' in the form

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$

(8 marks)

(iii) Hence explain why C is a hyperbola. *(1 mark)*

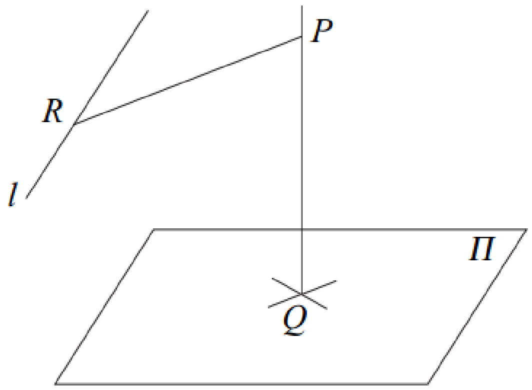
8 The point Q has position vector $\mathbf{q} = \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix}$, the plane Π has equation $\mathbf{r} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 36$,

and the line l has equation $\mathbf{r} = \begin{bmatrix} 20 \\ -8 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -7 \\ 5 \\ 3 \end{bmatrix}$.

(a) Show that Q lies in Π. *(1 mark)*

(b) Show also that l is parallel to Π. *(2 marks)*

(c) The diagram shows the point P, which lies on the normal to Π that passes through Q. The point R is the point on l which is closest to P, and $PQ = PR$.



Determine the coordinates of P. *(9 marks)*

January 2013

8 The four vertices of a parallelogram $ABCD$ have coordinates

$$A(1,\ 0,\ 2),\ B(3,\ -1,\ 5),\ C(7,\ 2,\ 4) \text{ and } D(5,\ 3,\ 1)$$

(a) (i) Find $\overrightarrow{AB} \times \overrightarrow{AD}$. *(3 marks)*

(ii) Show that the area of the parallelogram is $p\sqrt{10}$, where p is an integer to be found. *(2 marks)*

(b) The diagonals AC and BD of the parallelogram meet at the point M. The line L passes through M and is perpendicular to the plane $ABCD$.

Find an equation for the line L, giving your answer in the form $(\mathbf{r} - \mathbf{u}) \times \mathbf{v} = \mathbf{0}$. *(4 marks)*

(c) The plane Π is parallel to the plane $ABCD$ and passes through the point $Q(6,\ 5,\ 17)$.

(i) Find the coordinates of the point of intersection of the line L with the plane Π. *(6 marks)*

(ii) One face of a parallelepiped is $ABCD$ and the opposite face lies in the plane Π.

Find the volume of the parallelepiped. *(3 marks)*

June 2013

4 Two planes have equations

$$2x - 2y + z = 24$$

and

$$x + 3y + 4z = 8$$

They meet in a line L.

(a) Find Cartesian equations for the line L. *(5 marks)*

(b) The direction cosines of the line L are given by $\cos\alpha$, $\cos\beta$ and $\cos\gamma$.

(i) Find the exact value of each of the direction cosines. *(2 marks)*

(ii) Show that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$. *(3 marks)*