FP4: Matrix Algebra

Past Paper Questions
2006 - 2013

Name:

January 2006

2 The matrices $\mathbf{P}$ and $\mathbf{Q}$ are defined in terms of the constant k by

$$\mathbf{P} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & k \\ 5 & 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 5 & 4 & 1 \\ 3 & k & -1 \\ 7 & 3 & 2 \end{bmatrix}$$

(a) Express $\det \mathbf{P}$ and $\det \mathbf{Q}$ in terms of k. *(3 marks)*

(b) Given that $\det(\mathbf{PQ}) = 16$, find the two possible values of k. *(4 marks)*

June 2006

6 The matrices $\mathbf{P}$ and $\mathbf{Q}$ are given by

$$\mathbf{P} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & t & -2 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 1 & 1 & 1 \\ -7 & -1 & 5 \\ 11 & -1 & -7 \end{bmatrix}$$

where t is a real constant.

(a) Find the value of t for which $\mathbf{P}$ is singular. *(2 marks)*

(b) (i) Determine the matrix $\mathbf{R} = \mathbf{PQ}$, giving its elements in terms of t where appropriate. *(3 marks)*

(ii) Find the value of t for which $\mathbf{R} = k\mathbf{I}$, for some integer k. *(2 marks)*

(iii) Hence find the matrix $\mathbf{Q}^{-1}$. *(1 mark)*

(c) In the case when $t = -3$, describe the geometrical transformation with matrix $\mathbf{R}$. *(2 marks)*

January 2007

8 The matrix $\mathbf{P} = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 1 & 3 \\ -2 & 0 & a \end{bmatrix}$, where a is constant.

(a) (i) Determine $\det \mathbf{P}$ as a linear expression in a. *(2 marks)*

(ii) Evaluate $\det \mathbf{P}$ in the case when $a = 3$. *(1 mark)*

(iii) Find the value of a for which $\mathbf{P}$ is singular. *(2 marks)*

(b) The 3×3 matrix $\mathbf{Q}$ is such that $\mathbf{PQ} = 25\mathbf{I}$.

Without finding $\mathbf{Q}$:

(i) write down an expression for $\mathbf{P}^{-1}$ in terms of $\mathbf{Q}$; *(1 mark)*

(ii) find the value of the constant k such that $(\mathbf{PQ})^{-1} = k\mathbf{I}$; *(2 marks)*

(iii) determine the numerical value of $\det \mathbf{Q}$ in the case when $a = 3$. *(4 marks)*

June 2007

6 The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & t \end{bmatrix}$$

(a) Find, in terms of t, the matrices:

(i) **AB**; *(3 marks)*

(ii) **BA**. *(2 marks)*

(b) Explain why **AB** is singular for all values of t. *(1 mark)*

(c) In the case when $t = -2$, show that the transformation with matrix **BA** is the combination of an enlargement, E, and a second transformation, F. Find the scale factor of E and give a full geometrical description of F. *(6 marks)*

January 2008 (Only part a)

7 The non-singular matrix $\mathbf{M} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

(a) (i) Show that

$$\mathbf{M}^2 + 2\mathbf{I} = k\mathbf{M}$$

for some integer k to be determined. *(3 marks)*

(ii) By multiplying the equation in part (a)(i) by $\mathbf{M}^{-1}$, show that

$$\mathbf{M}^{-1} = a\mathbf{M} + b\mathbf{I}$$

for constants a and b to be found. *(3 marks)*

(b) (i) Determine the characteristic equation of **M** and show that **M** has a repeated eigenvalue, 1, and another eigenvalue, 2. *(6 marks)*

(ii) Give a full set of eigenvectors for each of these eigenvalues. *(5 marks)*

(iii) State the geometrical significance of each set of eigenvectors in relation to the transformation with matrix **M**. *(3 marks)*

June 2008

3 The matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{bmatrix}$, where k is a constant.

Determine, in terms of k where appropriate:

(a) $\det \mathbf{A}$; *(2 marks)*

(b) $\mathbf{A}^{-1}$. *(5 marks)*

January 2009

2 The 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ are such that

$$\mathbf{AB} = \begin{bmatrix} 9 & 1 \\ 7 & 13 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 14 & 2 \\ 1 & 8 \end{bmatrix}$$

Without finding $\mathbf{A}$ and $\mathbf{B}$:

(a) find the value of $\det \mathbf{B}$, given that $\det \mathbf{A} = 10$; *(3 marks)*

(b) determine the 2×2 matrices $\mathbf{C}$ and $\mathbf{D}$ given by

$$\mathbf{C} = (\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}) \quad \text{and} \quad \mathbf{D} = (\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}})^{\mathrm{T}}$$

where $\mathbf{M}^{\mathrm{T}}$ denotes the transpose of matrix $\mathbf{M}$. *(3 marks)*

June 2009

1 Let $\mathbf{P} = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 2 & 6 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} k & 1 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$, where k is a constant.

(a) Determine the product matrix $\mathbf{PQ}$, giving its elements in terms of k where appropriate. *(3 marks)*

(b) Find the value of k for which $\mathbf{PQ}$ is singular. *(2 marks)*

January 2010

3 The matrices $\mathbf{A}$ and $\mathbf{B}$ are defined in terms of a real parameter t by

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & t & 4 \\ 3 & 2 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 15 & -4 & -1 \\ -2t & 4 & 2 \\ 17 & -4 & -3 \end{bmatrix}$$

(a) Find, in terms of t, the matrix $\mathbf{AB}$ and deduce that there exists a value of t such that $\mathbf{AB}$ is a scalar multiple of the 3×3 identity matrix $\mathbf{I}$. *(5 marks)*

(b) For this value of t, deduce $\mathbf{A}^{-1}$. *(2 marks)*

June 2010

2 Let $\mathbf{A} = \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 4-4x & 8 \\ 8x-4 & 4 \end{bmatrix}$.

(a) Find $\mathbf{AB}$ in terms of x. *(2 marks)*

(b) Show that $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \mathbf{C}$ for some value of x. *(5 marks)*

January 2011

4 The non-singular matrix $\mathbf{X} = \begin{bmatrix} 3 & 1 & -3 \\ 2 & 4 & 3 \\ -4 & 2 & -1 \end{bmatrix}$.

(a) (i) Show that $\mathbf{X}^2 - \mathbf{X} = k\mathbf{I}$ for some integer k. *(3 marks)*

(ii) Hence show that $\mathbf{X}^{-1} = \frac{1}{20}(\mathbf{X} - \mathbf{I})$. *(2 marks)*

(b) The 3×3 matrix $\mathbf{Y}$ has inverse $\mathbf{Y}^{-1} = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 0 & -10 \\ 0 & 20 & 0 \end{bmatrix}$.

Without finding $\mathbf{Y}$, determine the matrix $(\mathbf{XY})^{-1}$. *(3 marks)*

June 2011

1 The matrices $\mathbf{A}$ and $\mathbf{B}$ are given in terms of p by

$$\mathbf{A} = \begin{bmatrix} 1 & p & 4 \\ -3 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} p & 1 & 5 \\ 9 & p & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

(a) Find each of $\det \mathbf{A}$ and $\det \mathbf{B}$ in terms of p. *(3 marks)*

(b) Without finding $\mathbf{AB}$, determine all values of p for which $\mathbf{AB}$ is singular. *(3 marks)*

January 2012

4 Let $\mathbf{X} = \begin{bmatrix} 3 & x \\ -1 & 7 \end{bmatrix}$.

(a) Determine $\mathbf{X}\mathbf{X}^{\mathrm{T}}$. *(2 marks)*

(b) Show that $\mathrm{Det}(\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}^{\mathrm{T}}\mathbf{X}) \leqslant 0$ for all real values of x. *(4 marks)*

(c) Find the value of x for which the matrix $(\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}^{\mathrm{T}}\mathbf{X})$ is singular. *(1 mark)*

7 The matrix $\mathbf{A} = \begin{bmatrix} k & 1 & 2 \\ 2 & k & 1 \\ 1 & 2 & k \end{bmatrix}$, where k is a real constant.

(a) (i) Show that there is a value of k for which

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = m\mathbf{I}$$

where m is a rational number to be determined and $\mathbf{I}$ is the 3×3 identity matrix. *(6 marks)*

(ii) Deduce the inverse matrix, $\mathbf{A}^{-1}$, of $\mathbf{A}$ for this value of k. *(1 mark)*

(b) (i) Find $\det \mathbf{A}$ in terms of k. *(2 marks)*

(ii) In the case when $\mathbf{A}$ is singular, find the integer value of k and show that there are no other possible real values of k. *(3 marks)*

(iii) Find the value of k for which $\lambda = 7$ is a real eigenvalue of $\mathbf{A}$. *(2 marks)*

January 2013 (not d)

4 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

(a) Given that $\mathbf{A}^2 = \begin{bmatrix} p & -2 & -4 \\ 5 & 6 & 4 \\ 10 & q & 9 \end{bmatrix}$, find the value of p and the value of q. *(2 marks)*

(b) Given that $\mathbf{A}^3 - 6\mathbf{A}^2 + 11\mathbf{A} - 6\mathbf{I} = \mathbf{0}$, prove that

$$\mathbf{A}^{-1} = \frac{1}{6}(\mathbf{A}^2 - 6\mathbf{A} + 11\mathbf{I})$$ *(2 marks)*

(c) Given that $\mathbf{A}^{-1} = \frac{1}{6}\begin{bmatrix} r & -2 & 2 \\ -1 & 5 & -2 \\ -2 & s & 2 \end{bmatrix}$, find the value of r and the value of s.

(2 marks)

(d) Hence, or otherwise, find the solution of the system of equations

$$\begin{aligned} x \quad\quad - z &= k \\ x + 2y + z &= 5 \\ 2x + 2y + 3z &= 7 \end{aligned}$$

giving your answers in terms of k. *(3 marks)*

June 2013

7 The 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy

$$\mathbf{AB} = \begin{bmatrix} k & 8 & 1 \\ 1 & 1 & 0 \\ 1 & 4 & 0 \end{bmatrix}, \text{ where } \mathbf{A} = \begin{bmatrix} k & 6 & 8 \\ 0 & 1 & 2 \\ -3 & 4 & 8 \end{bmatrix}$$

and k is a constant.

(a) Show that $\mathbf{AB}$ is non-singular. *(1 mark)*

(b) Find $(\mathbf{AB})^{-1}$ in terms of k. *(5 marks)*

(c) Find $\mathbf{B}^{-1}$. *(4 marks)*