Core 3: Numerical Methods

Past Paper Questions
2006 - 2013

Name:

Numerical integration

The trapezium rule: $\int_a^b y \, dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \frac{b-a}{n}$

The mid-ordinate rule: $\int_a^b y \, dx \approx h(y_{\frac{1}{2}} + y_{\frac{3}{2}} + \ldots + y_{n-\frac{3}{2}} + y_{n-\frac{1}{2}})$, where $h = \frac{b-a}{n}$

Simpson's rule: $\int_a^b y \, dx \approx \frac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\}$

where $h = \frac{b-a}{n}$ and n is even

January 2006

2 Use Simpson's rule with 5 ordinates (4 strips) to find an approximation to

$$\int_1^3 \frac{1}{\sqrt{1+x^3}}\,dx$$

giving your answer to three significant figures. *(4 marks)*

6 [**Figure 1**, printed on the insert, is provided for use in this question.]

The curve $y = x^3 + 4x - 3$ intersects the x-axis at the point A where $x = \alpha$.

(a) Show that α lies between 0.5 and 1.0. *(2 marks)*

(b) Show that the equation $x^3 + 4x - 3 = 0$ can be rearranged into the form $x = \dfrac{3 - x^3}{4}$. *(1 mark)*

(c) (i) Use the iteration $x_{n+1} = \dfrac{3 - x_n^{\,3}}{4}$ with $x_1 = 0.5$ to find x_3, giving your answer to two decimal places. *(3 marks)*

(ii) The sketch on **Figure 1** shows parts of the graphs of $y = \dfrac{3 - x^3}{4}$ and $y = x$, and the position of x_1.

On **Figure 1**, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis. *(3 marks)*

June 2006

1 The curve $y = x^3 - x - 7$ intersects the x-axis at the point where $x = \alpha$.

(a) Show that α lies between 2.0 and 2.1. *(2 marks)*

(b) Show that the equation $x^3 - x - 7 = 0$ can be rearranged in the form $x = \sqrt[3]{x + 7}$. *(1 mark)*

(c) Use the iteration $x_{n+1} = \sqrt[3]{x_n + 7}$ with $x_1 = 2$ to find the values of x_2, x_3 and x_4, giving your answers to three significant figures. *(3 marks)*

6 (a) Use the mid-ordinate rule with four strips to find an estimate for $\int_1^5 \ln x \, dx$, giving your answer to three significant figures. *(3 marks)*

(b) (i) Given that $y = x \ln x$, find $\frac{dy}{dx}$. *(2 marks)*

(ii) Hence, or otherwise, find $\int \ln x \, dx$. *(2 marks)*

(iii) Find the exact value of $\int_1^5 \ln x \, dx$. *(2 marks)*

January 2007

1 Use the mid-ordinate rule with four strips of equal width to find an estimate for $\int_1^5 \frac{1}{1 + \ln x} dx$, giving your answer to three significant figures. *(4 marks)*

June 2007

4 [**Figure 1**, printed on the insert, is provided for use in this question.]

(a) Use Simpson's rule with 5 ordinates (4 strips) to find an approximation to $\int_1^2 3^x \, dx$, giving your answer to three significant figures. *(4 marks)*

(b) The curve $y = 3^x$ intersects the line $y = x + 3$ at the point where $x = \alpha$.

(i) Show that α lies between 0.5 and 1.5. *(2 marks)*

(ii) Show that the equation $3^x = x + 3$ can be rearranged into the form

$$x = \frac{\ln(x+3)}{\ln 3}$$ *(2 marks)*

(iii) Use the iteration $x_{n+1} = \frac{\ln(x_n + 3)}{\ln 3}$ with $x_1 = 0.5$ to find x_3 to two significant figures. *(2 marks)*

(iv) The sketch on **Figure 1** shows part of the graphs of $y = \frac{\ln(x+3)}{\ln 3}$ and $y = x$, and the position of x_1.

On **Figure 1**, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis. *(2 marks)*

January 2008

3 The equation

$$x + (1 + 3x)^{\frac{1}{4}} = 0$$

has a single root, α.

(a) Show that α lies between -0.33 and -0.32. *(2 marks)*

(b) Show that the equation $x + (1 + 3x)^{\frac{1}{4}} = 0$ can be rearranged into the form

$$x = \tfrac{1}{3}(x^4 - 1)$$ *(2 marks)*

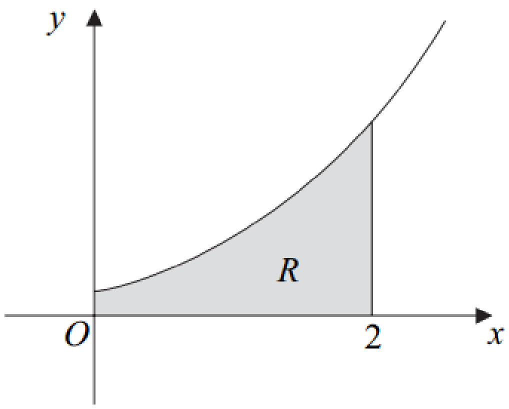
(c) Use the iteration $x_{n+1} = \dfrac{(x_n{}^4 - 1)}{3}$ with $x_1 = -0.3$ to find x_4, giving your answer to three significant figures. *(3 marks)*

6 (a) Sketch the curve with equation $y = \operatorname{cosec} x$ for $0 < x < \pi$. *(2 marks)*

(b) Use the mid-ordinate rule with four strips to find an estimate for $\int_{0.1}^{0.5} \operatorname{cosec} x \, dx$, giving your answer to three significant figures. *(4 marks)*

June 2008

6 The diagram shows the curve with equation $y = (e^{3x} + 1)^{\frac{1}{2}}$ for $x \geqslant 0$.



Use the mid-ordinate rule with four strips to find an estimate for $\int_0^2 (e^{3x} + 1)^{\frac{1}{2}} \, dx$, giving your answer to three significant figures. *(4 marks)*

January 2009

1 Use Simpson's rule with 5 ordinates (4 strips) to find an approximation to $\int_1^9 \dfrac{1}{1 + \sqrt{x}} \, dx$, giving your answer to three significant figures. *(4 marks)*

1 (a) The curve with equation

$$y = \frac{\cos x}{2x+1}, \qquad x > -\frac{1}{2}$$

intersects the line $y = \frac{1}{2}$ at the point where $x = \alpha$.

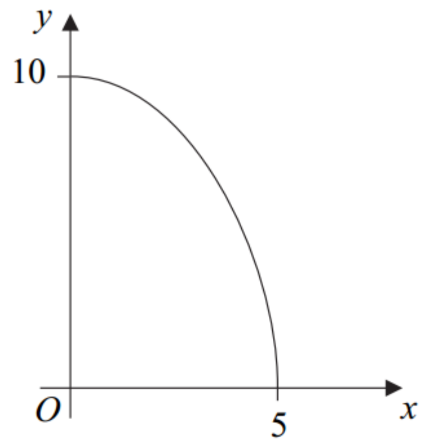
(i) Show that α lies between 0 and $\frac{\pi}{2}$. *(2 marks)*

(ii) Show that the equation $\frac{\cos x}{2x+1} = \frac{1}{2}$ can be rearranged into the form

$$x = \cos x - \frac{1}{2}$$ *(1 mark)*

(iii) Use the iteration $x_{n+1} = \cos x_n - \frac{1}{2}$ with $x_1 = 0$ to find x_3, giving your answer to three decimal places. *(2 marks)*

6 The diagram shows the curve with equation $y = \sqrt{100 - 4x^2}$, where $x \geqslant 0$.



Use the mid-ordinate rule with five strips of equal width to find an estimate for $\int_0^5 \sqrt{100 - 4x^2}\,dx$, giving your answer to three significant figures. *(4 marks)*

January 2010

2 [**Figure 1**, printed on the insert, is provided for use in this question.]

The equation $\sin^{-1} x = \frac{1}{4}x + 1$ can be rewritten as $x = \sin\left(\frac{1}{4}x + 1\right)$.

(i) Use the iteration $x_{n+1} = \sin\left(\frac{1}{4}x_n + 1\right)$ with $x_1 = 0.5$ to find the values of x_2 and x_3, giving your answers to three decimal places. *(2 marks)*

(ii) The sketch on **Figure 1** shows parts of the graphs of $y = \sin\left(\frac{1}{4}x + 1\right)$ and $y = x$, and the position of x_1.

On **Figure 1**, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis. *(2 marks)*

5 (a) Use the mid-ordinate rule with four strips to find an estimate for $\int_0^{12} \ln(x^2 + 5)\,dx$, giving your answer to three significant figures. *(4 marks)*

June 2010

1 The curve $y = 3^x$ intersects the curve $y = 10 - x^3$ at the point where $x = \alpha$.

(a) Show that α lies between 1 and 2. *(2 marks)*

(b) (i) Show that the equation $3^x = 10 - x^3$ can be rearranged into the form $x = \sqrt[3]{10 - 3^x}$. *(1 mark)*

(ii) Use the iteration $x_{n+1} = \sqrt[3]{10 - 3^{x_n}}$ with $x_1 = 1$ to find the values of x_2 and x_3, giving your answers to three decimal places. *(2 marks)*

4 (a) Use Simpson's rule with 7 ordinates (6 strips) to find an approximation to $\int_{0.5}^{2} \frac{x}{1 + x^3}\,dx$, giving your answer to three significant figures. *(4 marks)*

(b) Find the exact value of $\int_0^1 \frac{x^2}{1 + x^3}\,dx$. *(4 marks)*

January 2011

2 A curve is defined by the equation $y = (x^2 - 4)\ln(x+2)$ for $x \geqslant 3$.

The curve intersects the line $y = 15$ at a single point, where $x = \alpha$.

(a) Show that α lies between 3.5 and 3.6. *(2 marks)*

(b) Show that the equation $(x^2 - 4)\ln(x+2) = 15$ can be arranged into the form

$$x = \pm\sqrt{4 + \frac{15}{\ln(x+2)}}$$ *(2 marks)*

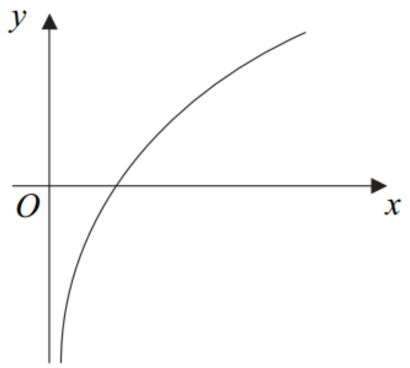
(c) Use the iteration

$$x_{n+1} = \sqrt{4 + \frac{15}{\ln(x_n+2)}}$$

with $x_1 = 3.5$ to find the values of x_2 and x_3, giving your answers to three decimal places. *(2 marks)*

June 2011

1 The diagram shows the curve with equation $y = \ln(6x)$.



Use Simpson's rule with 6 strips (7 ordinates) to find an estimate for $\int_1^7 \ln(6x)\,dx$, giving your answer to three significant figures. *(4 marks)*

3 The curve $y = \cos^{-1}(2x-1)$ intersects the curve $y = e^x$ at a single point where $x = \alpha$.

(a) Show that α lies between 0.4 and 0.5. *(2 marks)*

(b) Show that the equation $\cos^{-1}(2x-1) = e^x$ can be written as $x = \frac{1}{2} + \frac{1}{2}\cos(e^x)$. *(1 mark)*

(c) Use the iteration $x_{n+1} = \frac{1}{2} + \frac{1}{2}\cos(e^{x_n})$ with $x_1 = 0.4$ to find the values of x_2 and x_3, giving your answers to three decimal places. *(2 marks)*

January 2012

1 (a) Use Simpson's rule with 7 ordinates (6 strips) to find an estimate for $\int_0^3 4^x \, dx$.

(4 marks)

(b) A curve is defined by the equation $y = 4^x$. The curve intersects the line $y = 8 - 2x$ at a single point where $x = \alpha$.

(i) Show that α lies between 1.2 and 1.3. *(2 marks)*

(ii) The equation $4^x = 8 - 2x$ can be rearranged into the form $x = \dfrac{\ln(8 - 2x)}{\ln 4}$.

Use the iterative formula $x_{n+1} = \dfrac{\ln(8 - 2x_n)}{\ln 4}$ with $x_1 = 1.2$ to find the values of x_2 and x_3, giving your answers to three decimal places. *(2 marks)*

June 2012

1 Use the mid-ordinate rule with four strips to find an estimate for $\int_{0.4}^{1.2} \cot(x^2) \, dx$, giving your answer to three decimal places. *(4 marks)*

2 For $0 < x \leqslant 2$, the curves with equations $y = 4 \ln x$ and $y = \sqrt{x}$ intersect at a single point where $x = \alpha$.

(a) Show that α lies between 0.5 and 1.5. *(2 marks)*

(b) Show that the equation $4 \ln x = \sqrt{x}$ can be rearranged into the form

$$x = e^{\left(\frac{\sqrt{x}}{4}\right)}$$

(1 mark)

(c) Use the iterative formula

$$x_{n+1} = e^{\left(\frac{\sqrt{x_n}}{4}\right)}$$

with $x_1 = 0.5$ to find the values of x_2 and x_3, giving your answers to three decimal places. *(2 marks)*

(d) **Figure 1**, on the opposite page, shows a sketch of parts of the graphs of $y = e^{\left(\frac{\sqrt{x}}{4}\right)}$ and $y = x$, and the position of x_1.

On **Figure 1**, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis. *(2 marks)*

January 2013

1 (a) Show that the equation $x^3 - 6x + 1 = 0$ has a root α, where $2 < \alpha < 3$. *(2 marks)*

(b) Show that the equation $x^3 - 6x + 1 = 0$ can be rearranged into the form

$$x^2 = 6 - \frac{1}{x}$$ *(1 mark)*

(c) Use the recurrence relation $x_{n+1} = \sqrt{6 - \frac{1}{x_n}}$, with $x_1 = 2.5$, to find the value of x_3, giving your answer to four significant figures. *(2 marks)*

2 (a) Use Simpson's rule, with five ordinates (four strips), to calculate an estimate for

$$\int_0^4 \frac{x}{x^2 + 2}\,\mathrm{d}x$$

Give your answer to four significant figures. *(4 marks)*

(b) Show that the exact value of $\int_0^4 \frac{x}{x^2 + 2}\,\mathrm{d}x$ is $\ln k$, where k is an integer. *(5 marks)*

June 2013

3 (a) The equation $\mathrm{e}^{-x} - 2 + \sqrt{x} = 0$ has a single root, α.

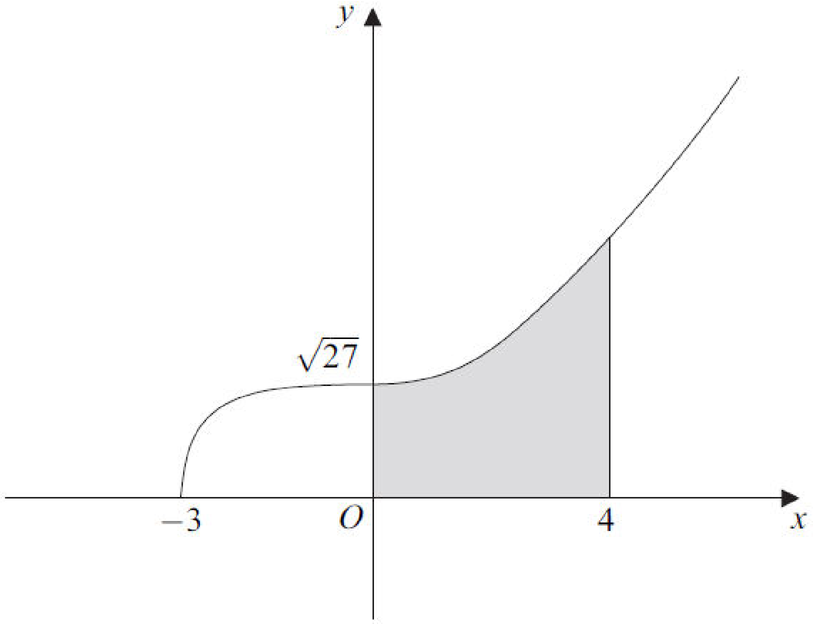
Show that α lies between 3 and 4. *(2 marks)*

(b) Use the recurrence relation $x_{n+1} = (2 - \mathrm{e}^{-x_n})^2$, with $x_1 = 3.5$, to find x_2 and x_3, giving your answers to three decimal places. *(2 marks)*

(c) The diagram on the opposite page shows parts of the graphs of $y = (2 - \mathrm{e}^{-x})^2$ and $y = x$, and a position of x_1.

On the diagram, draw a staircase or cobweb diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis. *(2 marks)*

5 The diagram shows a sketch of the graph of $y = \sqrt{27 + x^3}$.



(a) The area of the shaded region, bounded by the curve, the x-axis and the lines $x = 0$ and $x = 4$, is given by $\int_0^4 \sqrt{27 + x^3}\, dx$.

Use the mid-ordinate rule with **five** strips to find an estimate for this area. Give your answer to three significant figures. *(4 marks)*

(b) With the aid of a diagram, explain whether the mid-ordinate rule applied in part **(a)** gives an estimate which is smaller than or greater than the area of the shaded region. *(2 marks)*

Inserts to be used with Core 3: Numerical Methods Past Paper Question 2006 - 2013

January 2006

Figure 1 (for Question 6)

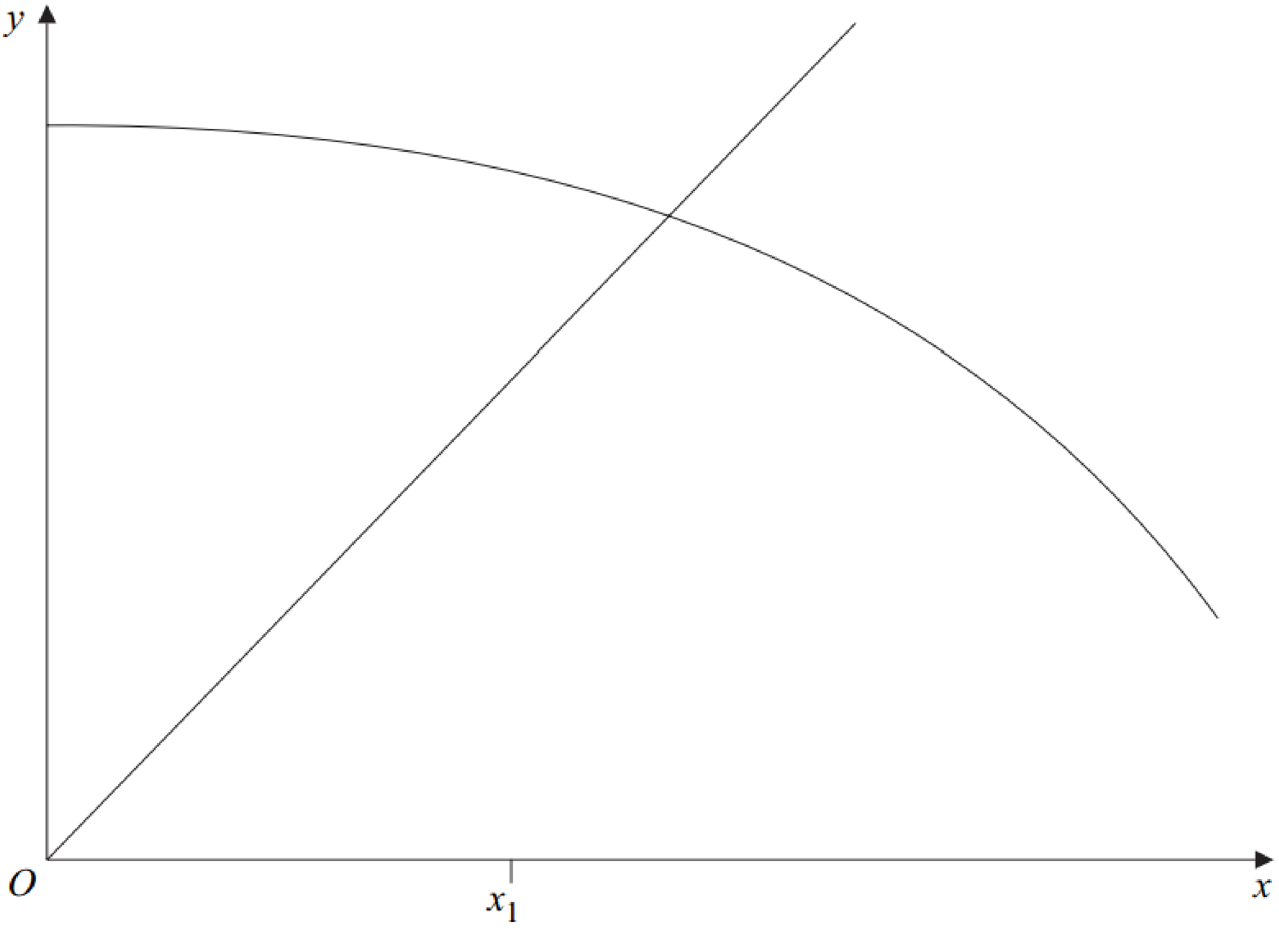


Figure 1 (for use in Question 4)

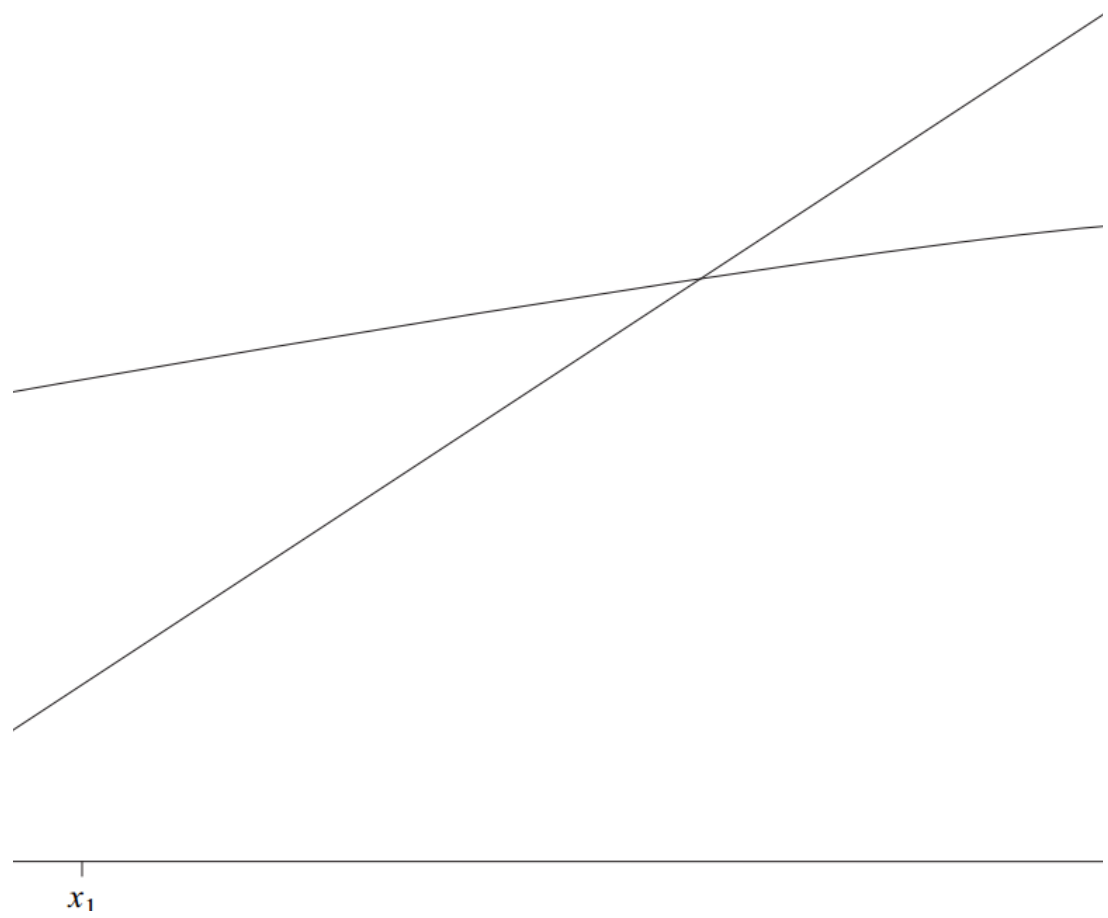
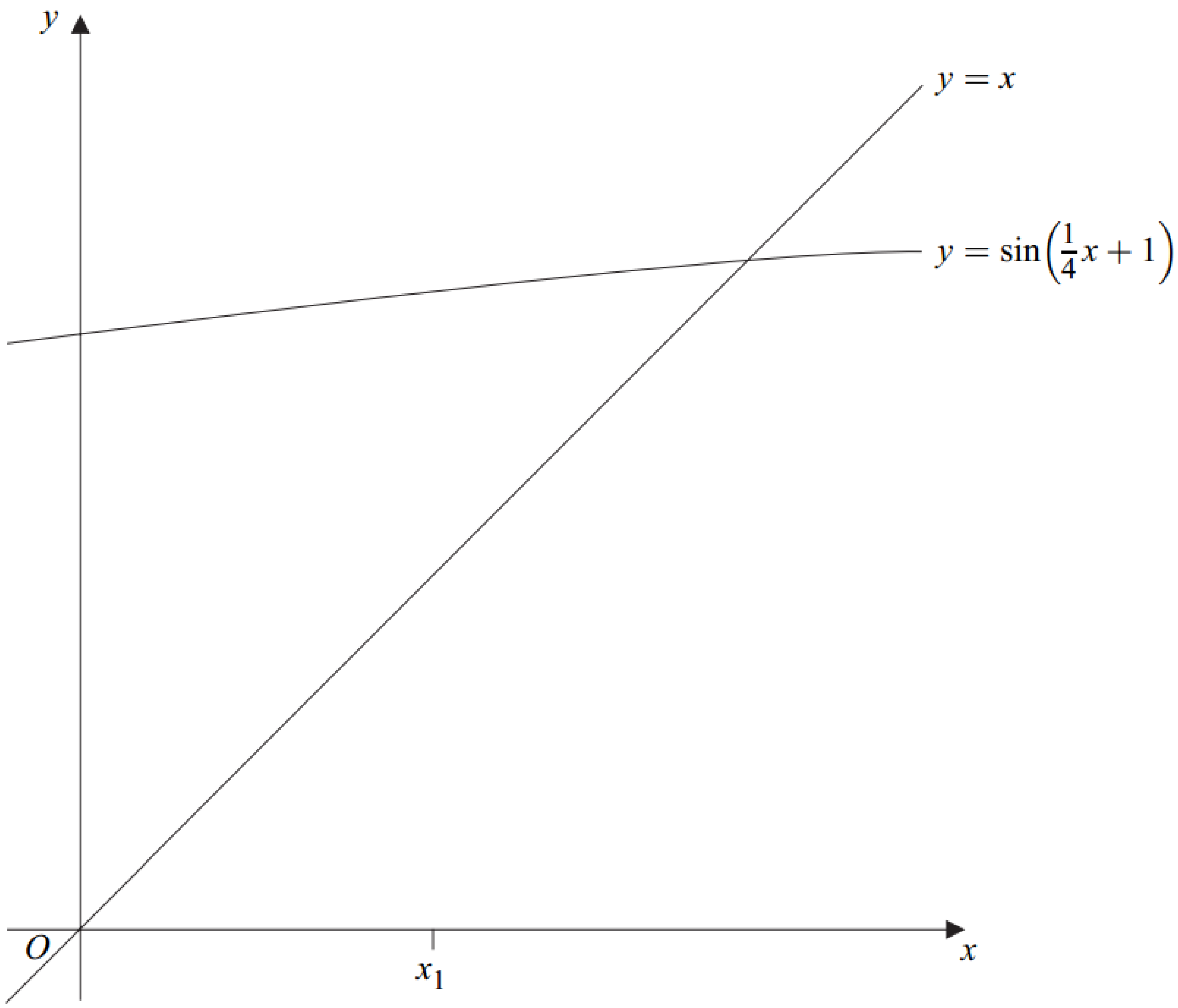


Figure 1 (for use in Question 2)



June 2012

Figure 1