(a) Show that the equation

$$4\sinh x + e^x = 5$$

can be expressed as

$$3e^{2x} - 5e^x - 2 = 0.$$ *(3 marks)*

(b) Hence solve, for real x,

$$4\sinh x + e^x = 5,$$

giving your answer as a natural logarithm. *(4 marks)*

2(a)	Substituting $\sinh x = \frac{e^x - e^{-x}}{2}$	M1		
	$2(e^x - e^{-x}) + e^x = 5$	A1		for $4(e^x - e^{-x}) + e^x = 5$ M1A0
	$3e^{2x} - 5e^x - 2 = 0$	A1	3	
(b)	$(3e^x + 1)(e^x - 2) = 0$	M1A1		
	$e^x = -\frac{1}{3}$ or 2	A1F		
	$x = \ln 2 \quad \left(e^x \neq -\frac{1}{3}\right)$	A1F	4	Do not award this mark if $\ln\left(-\frac{1}{3}\right)$ is also given as an answer
	Total		**7**	

(a) Given that

$$\cosh(x + y) \equiv \cosh x \cosh y + \sinh x \sinh y,$$

write down the expansion of $\cosh(x - y)$. *(1 mark)*

(b) The positive numbers x and y, where $x > y$, satisfy the equations

$$\cosh x \cosh y = 2.8,$$

$$\sinh x \sinh y = 0.2.$$

(i) Show that

$$x + y = \ln(3 + 2\sqrt{2}),$$

and find a corresponding result for $(x - y)$. *(5 marks)*

(ii) Hence show that

$$x = \frac{1}{2}\ln(15 + 10\sqrt{2}).$$ *(2 marks)*

Q	Solution	Marks	Total	Comments
4 (a)	$\cosh(x-y) \equiv \cosh x \cosh y - \sinh x \sinh y$	B1	1	accept =
(b)(i)	$\cosh(x+y) = 2.8 + 0.2$	M1		
	$x+y = \ln\left(3+\sqrt{3^2-1}\right)$	M1		
	$= \ln\left(3+2\sqrt{2}\right)$	A1		AG Ignore ± sign
	$x-y = \ln\left(2.6+\sqrt{2.6^2-1}\right)$	M1		
	$= \ln 5$	A1	5	
(ii)	$x = \frac{1}{2}\ln\left(15+10\sqrt{2}\right)$	M1A1	2	AG If numerical methods used earlier, allow final A1
	Total		**8**	

(a) Evaluate:

(i) $\int \cosh^2 x \, dx$; *(3 marks)*

(ii) $\int x \cosh x \, dx$. *(3 marks)*

(b) A curve C is given parametrically by the equations

$$x = \cosh t + t, \qquad y = \cosh t - t.$$

Express

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

in terms of $\cosh t$. *(5 marks)*

Q	Solution	Marks	Total	Comments
4 (a)(i)	$\int \cosh^2 x \, dx = \int \frac{1}{2}(1+\cosh 2x) dx$	M1A1		
	$= \frac{x}{2} + \frac{\sinh 2x}{4} (+c)$	A1F	3	
(ii)	$\int x\cosh x \, dx = x\sinh x - \int \sinh x \, dx$	M1A1		if $\int \sinh x$ is given as $-\cosh x$, penalise once only if consistent
	$= x\sinh x - \cosh x (+c)$	A1F	3	
(b)	$\dot{x} = \sinh t + 1, \quad \dot{y} = \sinh t - 1$	B1		
	$\dot{x}^2 + \dot{y}^2 = (\sinh t + 1)^2 + (\sinh t - 1)^2$	M1		
	$= \sinh^2 t + 2\sinh t + 1 + \sinh^2 t - 2\sinh t + 1$	A1F		
	Use of $\cosh^2 t - \sinh^2 t = 1$	m1		
	$= 2\cosh^2 t$	A1F	5	

(a) Use the identity

$$\tanh^{-1} x \equiv \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

to show that

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}.$$

(4 marks)

(b) (i) Use integration by parts to show that

$$\int \tanh^{-1} x \, dx = x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2) + c,$$

where c is a constant. *(4 marks)*

(ii) Hence evaluate

$$\int_0^{\frac{1}{3}} \tanh^{-1} x \, dx$$

giving your answer in the form

$$a \ln 2 + b \ln 3,$$

where a and b are rational numbers.

(6 marks)

Q	Solution	Marks	Total	Comments
7 (a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tanh^{-1}x\right)=\frac{1}{2}\left(\frac{1}{1+x}\right)+\frac{1}{2}\left(\frac{1}{1-x}\right)$	M1 A2,1		
	$=\frac{1}{1-x^2}$	A1	4	Shown
	Alternative to part (a)			
	$u=\frac{1+x}{1-x}$ $\quad \frac{\mathrm{d}u}{\mathrm{d}x}=\frac{2}{(1-x)^2}$	*(M1A1)*		
	$\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{1}{2u}$			
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-x}{2(1+x)}\ \frac{2}{(1-x)^2}$	*(A1√)*		
	$=\frac{1}{1-x^2}$	*(A1)*		CAO
(b)(i)	$\int \tanh^{-1}x = x\tanh^{-1}x-\int\frac{x}{1-x^2}\mathrm{d}x$	M1A1A1		
	$=x\tanh^{-1}x+\frac{1}{2}\ln\left(1-x^2\right)(+c)$	A1	4	
(ii)	$\int_0^{\frac{1}{3}}\tanh^{-1}x\,\mathrm{d}x=\left[x\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{1}{2}\ln\left(1-x^2\right)\right]_0^{\frac{1}{3}}$	M1		
	$=\frac{1}{3}\times\frac{1}{2}\ln 2+\frac{1}{2}\ln\frac{8}{9}$	A2,1		
	$=\frac{1}{6}\ln 2+\frac{1}{2}\ln 8-\frac{1}{2}\ln 9$	M1		Use of $\ln\frac{a}{b}=\ln a-\ln b$
	$=\frac{1}{6}\ln 2+\frac{3}{2}\ln 2-\ln 3$	M1		Use of $\ln a^n = n\ln a$
	$=\frac{5}{3}\ln 2-\ln 3$	A1√	6	
	Total		**14**	