FP2 - Hyperbolic functions

Challenge 1

(a) Show that the equation

$$4\sinh x + e^x = 5$$

can be expressed as

$$3e^{2x} - 5e^x - 2 = 0.$$

(3 marks)

(b) Hence solve, for real x,

$$4\sinh x + e^x = 5,$$

giving your answer as a natural logarithm. *(4 marks)*

Challenge 2

(a) Given that

$$\cosh(x + y) \equiv \cosh x \cosh y + \sinh x \sinh y,$$

write down the expansion of $\cosh(x - y)$. *(1 mark)*

(b) The positive numbers x and y, where $x > y$, satisfy the equations

$$\cosh x \cosh y = 2.8,$$

$$\sinh x \sinh y = 0.2.$$

(i) Show that

$$x + y = \ln(3 + 2\sqrt{2}),$$

and find a corresponding result for $(x - y)$. *(5 marks)*

(ii) Hence show that

$$x = \frac{1}{2}\ln(15 + 10\sqrt{2}).$$ *(2 marks)*

Challenge 3

(a) Evaluate:

(i) $\int \cosh^2 x \, dx$; *(3 marks)*

(ii) $\int x \cosh x \, dx$. *(3 marks)*

(b) A curve C is given parametrically by the equations

$$x = \cosh t + t, \qquad y = \cosh t - t.$$

Express

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

in terms of $\cosh t$. *(5 marks)*

Final Challenge

(a) Use the identity

$$\tanh^{-1} x \equiv \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

to show that

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}.$$

(4 marks)

(b) (i) Use integration by parts to show that

$$\int \tanh^{-1} x \, dx = x \tanh^{-1} x + \frac{1}{2}\ln(1-x^2) + c,$$

where c is a constant. *(4 marks)*

(ii) Hence evaluate

$$\int_0^{\frac{1}{3}} \tanh^{-1} x \, dx$$

giving your answer in the form

$$a \ln 2 + b \ln 3,$$

where a and b are rational numbers.

(6 marks)