C3 Logs and exponentials

Challenge 1

The point P lies on the curve with equation $y = \ln\left(\frac{1}{3}x\right)$. The x-coordinate of P is 3.

Find an equation of the normal to the curve at the point P in the form $y = ax + b$, where a and b are constants.

(Total 5 marks)

Challenge 2

(a) Sketch on one pair of axes the graphs of

$$y = 6 - x \text{ and } y = \ln x .$$ *(1 mark)*

(b) Hence state the number of roots of the equation

$$6 - x = \ln x .$$ *(1 mark)*

(c) By considering values of the function f, where

$$f(x) = 6 - x - \ln x ,$$

(i) show that the equation in part (b) has a root α such that

$$4 < \alpha < 5 ,$$ *(2 marks)*

(ii) determine whether α is closer to 4 or to 5. *(2 marks)*

Challenge 3

(a) (i) Draw on the same diagram sketches of the graphs with equations

$$y = 5e^{2x} \quad \text{and} \quad y = \frac{4}{x} \quad \text{for} \quad x > 0.$$

(2 marks)

(ii) Explain why this diagram shows that, for $x > 0$, the equation

$$5e^{2x} - \frac{4}{x} = 0$$

has just one root, α, and show that $0.3 < \alpha < 0.4$. *(2 marks)*

(b) Show, using calculus, that $y = 5e^{2x} - \dfrac{4}{x}$ is an increasing function of x for $x > 0$.

(3 marks)

(c) Show that the area of the region enclosed by the curve $y = 5e^{2x} - \dfrac{4}{x}$, the x-axis, and the lines $x = \frac{1}{2}$ and $x = 2$ can be expressed in the form

$$\frac{5}{2}\left(e^4 - e\right) - k \ln 2$$

for some positive integer k whose value is to be determined. *(5 marks)*

Final Challenge

(a) (i) Draw on the same diagram sketches of the graphs with equations

$$y = x - 2 \text{ and } y = 2\ln x \text{ for } x > 0$$ *(2 marks)*

(ii) Hence state the number of roots of the equation

$$x - 2 = 2\ln x, \qquad x > 0$$ *(1 mark)*

(b) The curve, C, with equation

$$y = x - 2 - 2\ln x, \qquad x > 0$$

has only one stationary point.

(i) Find $\frac{dy}{dx}$. *(2 marks)*

(ii) Show that the y-coordinate of the stationary point is $-\ln 4$. *(3 marks)*

(iii) Find $\frac{d^2y}{dx^2}$. *(2 marks)*

(iv) Hence show that the stationary point is a minimum. *(1 mark)*

(c) The vertical lines $x=6$ and $x=7$ meet the curve C at points P and Q respectively.

(i) Show that the y-coordinate of P is $4 - \ln 36$. *(2 marks)*

(ii) The area of the trapezium bounded by the lines PQ, $x=6$, $x=7$ and the x-axis is A square units. Show that

$$A = \frac{p}{2} - \ln q$$

stating the values of the positive integers p and q. *(3 marks)*