One of the roots of the cubic equation

$$x^3 + kx - 80 = 0,$$

where k is real, is $-2 + 4\text{i}$. Find the other two roots and the value of k. *(5 marks)*

Q	Solution	Marks	Total	Comments
1	$-2-4i$ is a root	B1		
	Sum of roots = 0 $\rightarrow -2+4i-2-4i+j=0$	M1		O.E. $\sum\alpha = k$, $\sum\alpha\beta = 80$ M0
	$j=4$	A1$\surd$		
	$f(4)=4^3+4k-80=0$	M1		
	$k=4$	A1$\surd$	5	k must be real
	Alternative $(x-(-2+4i))(x-(-2-4i)) = x^2+4x+20$	*(B1)*		
	either multiply or divide subsequently $\gamma=4 \quad k=4$	*(M1)* *(A1$\surd$A1$\surd$)*		
	Substitution of $-2+4i$ into equation unless complete and correct	*(M0)*		
	Total		**5**	

The cubic equation

$$x^3 + 2px^2 - 8 = 0, \qquad \text{where } p \text{ is real,}$$

has roots α, β and $\alpha + \beta$.

(a) Show that:

(i) $\alpha + \beta = -p$; *(2 marks)*

(ii) $\alpha\beta = -\dfrac{8}{p}$. *(2 marks)*

(b) Show that $p = 2$. *(5 marks)*

Q	Solution	Marks	Total	Comments
1 (a)(i)	$\alpha+\beta+(\alpha+\beta)=-2p$	M1		Must show some evidence of how the result is arrived at
	$\alpha+\beta=-p$	A1	2	
(ii)	Product of roots $\alpha\beta\gamma=\alpha\beta(\alpha+\beta)=8$	M1		
	$\alpha\beta=\dfrac{8}{-p}=-\dfrac{8}{p}$	A1	2	Must show some evidence of how the result is arrived at
(b)	$\alpha\beta+\beta\gamma+\gamma\alpha=\alpha\beta+\beta(\alpha+\beta)$ $+\alpha(\alpha+\beta)=0$	M1A1		
	$0=\alpha\beta+(\alpha+\beta)^2$	A1F		could be a mixture of α,β and p e.g. $-\dfrac{8}{p}-p\alpha-p\beta$
	$-\dfrac{8}{p}=-(-p)^2$	M1		for an equation in p only
	$p=2$	A1	5	AG
	Total		**9**	

The roots of the cubic equation

$$x^3 - 6x^2 + 4px - p^2 = 0$$

are $\beta - k$, β and $\beta + k$.

(a) Show that

(i) $\beta = 2$, *(2 marks)*

(ii) $k^2 = 4(3 - p)$, *(3 marks)*

(iii) $k^2 = 4 - \frac{1}{2}p^2$. *(2 marks)*

(b) Hence find

(i) the value of p, *(3 marks)*

(ii) the two non-real roots of the cubic equation giving your answers in the form $a + ib$, where a and b are real. *(3 marks)*

Q	Solution	Marks	Total	Comments
6 (a)(i)	$\beta - k + \beta + \beta + k = -(-6), \quad \beta = 2$	M1A1	2	AG
(ii)	$2(2+k) + 2(2-k) + (2+k)(2-k) = 4p$	M1A1		Allow M1 if there is a sign error
	$k^2 = 4(3-p)$	A1	3	AG
(iii)	$(2-k)2(2+k) = p^2$	M1		
	$k^2 = 4 - \frac{1}{2}p^2$	A1	2	AG
(b)(i)	Elimination of k, or substitution of 2 into cubic	M1		
	$p^2 - 8p + 16 = 0$	A1		
	$(p-4)^2 = 0, \quad p = 4$	A1F	3	
(ii)	$k = \pm 2\mathrm{i}$	M1A1		allow B1 for single answer $2 + 2\mathrm{i}$
	Roots $2 \pm 2\mathrm{i}$	A1F	3	f.t. if k is complex
	Total		**13**	

(a) (i) Express $(1 - \mathrm{i})(3 - \mathrm{i})$ in the form $a + \mathrm{i}b$, where a and b are real. *(2 marks)*

(ii) Show that

$$(1 - \mathrm{i})^3 = -2 - 2\mathrm{i}.$$ *(2 marks)*

(b) (i) Show that $1 - \mathrm{i}$ is a root of the cubic equation

$$z^3 + \mathrm{i}z^2 - (3 - \mathrm{i})z + 2(1 - \mathrm{i}) = 0.$$ *(3 marks)*

(ii) Given that the other two roots of the cubic equation are α and β, show that

$$\alpha + \beta = -1$$

and find $\alpha\beta$. *(3 marks)*

(c) Hence solve the cubic equation completely. *(3 marks)*

2 (a)(i)	$(1-\mathrm{i})(3-\mathrm{i}) = 2 - 4\mathrm{i}$	M1 A1	2	for attempt to multiply or use $\mathrm{i}^2 = -1$
(ii)	$(1-\mathrm{i})^3 = (1 - 3\mathrm{i} + 3\mathrm{i}^2 - \mathrm{i}^3) = -2 - 2\mathrm{i}$	M1 A1	2	for any complete method
(b)(i)	$(1-\mathrm{i})^3 + \mathrm{i}(1-\mathrm{i})^2 - (3-\mathrm{i})(1-\mathrm{i}) + 2 - 2\mathrm{i}$ = 0 shown	M1A1√ A1	3	CAO
(ii)	$\alpha + \beta + 1 - \mathrm{i} = -\mathrm{i} \quad \therefore \alpha + \beta = -1$ $\alpha\beta(1-\mathrm{i}) = -2(1-\mathrm{i})$ $\alpha\beta = -2$	B1 M1 A1	3	allow if sign error
(c)	Equation for α, β is $z^2 + z - 2 = 0$ Roots are $(1-\mathrm{i})$, 1, -2	M1A1√ A1√	3	
	Total		**13**	