FP4 - Vector Geometry Challenge

Challenge 1

(a) Find $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. *(2 marks)*

(b) The points P and Q lie on the lines

$$\mathbf{r} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \text{respectively.}$$

(i) Express the vector $\overrightarrow{PQ}$ in terms of s and t. *(2 marks)*

(ii) Hence find the scalar λ for which

$$\overrightarrow{PQ} = \lambda \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}.$$

(4 marks)

(c) Show that the vector $\overrightarrow{PQ}$ with the value of λ found in part (b)(ii) is perpendicular to both lines. *(2 marks)*

(d) Find the shortest possible distance PQ. *(2 marks)*

Challenge 2

The line l_1 has equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$.

The line l_2 has equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$.

(a) Show that the lines l_1 and l_2 intersect and find the coordinates of their point of intersection. *(5 marks)*

(b) (i) Show that the vector $\begin{bmatrix} 1 \\ 11 \\ -16 \end{bmatrix}$ is perpendicular to both l_1 and l_2. *(2 marks)*

(ii) Find the Cartesian equation of the plane containing the lines l_1 and l_2. *(3 marks)*

Challenge 3

The planes Π_1 and Π_2 have equations

$$x + 2y - z = 1$$

and $$x + 3y + z = 6$$

respectively.

(a) Verify that the point P, with coordinates $(1, 1, 2)$, lies on both planes. *(1 mark)*

(b) Find the equation of l, the line of intersection of Π_1 and Π_2, giving your answer in the form $\dfrac{x-\alpha}{p} = \dfrac{y-\beta}{q} = \dfrac{z-\gamma}{r}$. *(5 marks)*

(c) Find the acute angle between the line l and the line through P which is parallel to the y-axis. Give your answer to the nearest 0.1 of a degree. *(3 marks)*

Final Challenge

A line l has equation

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0},$$

where

$$\mathbf{a} = \begin{bmatrix} -4 \\ 10 \\ -4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

(a) (i) Show that the coordinates of any point on l may be written as

$$(-4 + 2\lambda,\ 10 - 3\lambda,\ -4 + \lambda),$$

where λ is a parameter. *(3 marks)*

(ii) Hence find the coordinates of the point N on l such that ON, where O is the origin, is perpendicular to l. *(4 marks)*

(b) Find the equation of the line through N which is perpendicular to both l and ON, giving your answer in the form $(\mathbf{r} - \mathbf{c}) \times \mathbf{d} = \mathbf{0}$. *(4 marks)*